Path-Based Algorithms to Solve C-Logit Stochastic User Equilibrium Assignment Problem

Xiangdong Xu, Anthony Chen, Zhong Zhou, and Shlomo Bekhor

This paper develops path-based algorithms to solve the C-logit stochastic user equilibrium (SUE) problem on the basis of an adaptation of the gradient projection method. The algorithms' strategies for step size determination differ. Three strategies are investigated: (*a*) predetermined step size, (*b*) Armijo line search, and (*c*) self-adaptive line search. The algorithms are tested on the well-known Winnipeg (Manitoba, Canada) network. Two sets of experiments are conducted: (*a*) a computational comparison of different line search strategies and (*b*) the impact of different modeling specifications for route overlapping (a flow-independent or a flow-dependent commonality factor). The results indicate that the path-based algorithm with the self-adaptive step size strategy performs better than the other step size strategies. The paper shows that, depending on the model parameters, particularly the commonality factor parameter, the C-logit SUE flows may be quite different from the multinomial logit SUE flows.

The stochastic user equilibrium (SUE) model is well known in the literature. It relaxes the perfect information assumption of the deterministic user equilibrium model by incorporating a random error term in the route cost function to simulate travelers' imperfect perceptions of travel times. Route choice models, under this approach, can have different specifications according to the modeling assumptions on the random error term. The two commonly used random error terms are Gumbel and normal distributions, corresponding to the logit-based and probit-based route choice models, respectively (*1, 2*). The logit-based route choice model has a closed-form probability expression, and the equivalent mathematical programming (MP) formulation can be formulated with an entropy-type model for the logit-based SUE problem (*3*). The drawbacks of the logit model are: (*a*) an inability to account for overlapping (or correlation) between routes and (*b*) an inability to account for perception variance with respect to trips of different lengths. These two drawbacks stem from the underlying assumptions that the random error terms are independently and identically distributed with the same and fixed variance (*4*). The probit-based route choice model does not have such drawbacks because it handles the overlapping and identical variance problems between routes by allowing covariance between the random error terms for pairs of routes. However, the probit model does not have a closed-form solution, and it is computationally burdensome when the choice set contains more than a handful of routes. As a result of its lack of a closed-form probability expression, solving the probit-based route choice model requires either Monte Carlo simulation (*5*), Clark's approximation method (*6*), or a numerical method (*7*).

To overcome the deficiencies of the multinomial logit (MNL) route choice model, some analytical closed-form extensions have been proposed in the literature. One group of extensions is based on the theory of generalized extreme value; this group includes the cross-nested logit model (*8*), the paired combinatorial logit model (*8*), and the generalized nested logit model (*9*). The generalized extreme value models capture the similarity between routes through the random error component of the utility function. The model structure is generally a two-level tree structure, which allows an alternative (route) to belong to more than one nest (a nest here is a link in the cross-nested logit and generalized nested logit models or a pair of routes in the paired combinatorial logit model). The choice probability is calculated according to the two-level tree structure through the use of marginal and conditional probabilities. Equivalent MP formulations for all three models were given by Bekhor and Prashker (*8, 9*). For a more detailed review of these generalized extreme value route choice models used in the SUE problem, please refer to Prashker and Bekhor (*10*) and Prato (*11*).

Another group of extensions modify the deterministic (or systematic) term of the utility function to account for the overlapping problem while still retaining the single-level tree structure of the MNL model. The models in this group include the C-logit model, the implicit availability–perception model, and the path-size logit model (*12–14*). All three models add a correction term to the deterministic term of the utility function to adjust the choice probability; however, the interpretation of each model is different. The C-logit model uses the commonality factor (CF) to penalize the coupling routes, while the implicit availability–perception and path-size logit models both use a logarithmic correction term to modify the utility (hence, the choice probability).

Of the above extensions, the C-logit model has been adopted in many applications, such as the path flow estimator that estimates an origin–destination (O-D) trip table from traffic counts (M. G. H. Bell, unpublished lecture notes on transportation network analysis, 1998), microscopic traffic simulation (e.g., AIMSUN), and network design problems (*15*). This popularity is attributed to the C-logit

X. Xu, School of Transportation, Southeast University, Nanjing 210096, China, and Department of Civil and Environmental Engineering, Utah State University, Logan, UT 84322-4110. A. Chen, Department of Civil and Environmental Engineering, Utah State University, Logan, UT 84322-4110. Z. Zhou, Citilabs, 316 Williams Street, Tallahassee, FL 32303. S. Bekhor, Faculty of Civil and Environmental Engineering, Technion–Israel Institute of Technology, Haifa 32000, Israel. Corresponding author: A. Chen, anthony.chen@usu.edu.

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model's analytical closed-form probability expression, its relatively lower calibration effort, and its sound rational behavior, consistent with random utility theory. Recently, Zhou et al. developed equivalent mathematical formulations of the SUE assignment problem corresponding to the C-logit route choice model (*16*). However, there has been little study of the practical implications of using the C-logit SUE model on the assignment results and even less development of appropriate algorithms for their solution. The efficient algorithmic development for solving the C-logit SUE model is quite important for the model implementation in large-scale networks. In addition, the effect of different modeling specifications for route overlapping on the equilibrium patterns is of interest to modelers attempting to develop better route choice models. This paper develops new algorithms for solving the C-logit SUE problem based on the adaptation of the gradient projection (GP) method and illustrates the application of the C-logit SUE model on real networks.

Early algorithms developed to solve the logit-based SUE problem were link based [e.g., Maher (*17*)]. These link-based algorithms do not require path storage and often use Dial's STOCH algorithm or Bell's alternative as the stochastic loading step (*1, 18*). Path-based algorithms require explicit path storage to directly compute the logit route choice probabilities. Damberg et al. developed a path-based algorithm based on the disaggregated simplicial decomposition algorithm to solve the MNL SUE problem (*19*). Bekhor and Toledo compared path-based algorithms for the MNL SUE problem, and showed that the disaggregated simplicial decomposition algorithm is superior to the path-based method of successive averages (MSA) algorithm (*20*). However, algorithms for solving the SUE problem with extended logit models, such as C-logit, are not fully developed.

The purpose of this paper is to investigate the C-logit SUE traffic assignment model and solution algorithms. Depending on the modeling specification for route overlapping, the CF could be flow independent or flow dependent, corresponding to the length-based and congestion-based models. The contributions of this paper are twofold:

1. Algorithmic development. Path-based algorithms are developed to solve the C-logit SUE models based on an adaptation of the GP method. The algorithms have different strategies for determination of step size. Three strategies are investigated: (*a*) predetermined step size, (*b*) Armijo line search, and (*c*) self-adaptive line search. Their computational performance is compared using real networks. Further, the robustness and sensitivity of the self-adaptive step size strategy implemented in the GP algorithm is examined.
2. Modeling specification for route overlapping. The effect of different modeling specifications for route overlapping (flow-independent or flow-dependent CF) on the equilibrium flow patterns at different levels is examined. Specifically, the paper explores how the route-overlapping considerations adjust the choice probabilities for routes that are coupled with other routes. In addition, the impact of route overlapping considerations on the equilibrium route and link flow patterns, as well as route-based network analysis, is investigated.

The remainder of the paper is organized as follows: (*a*) a presentation of the C-logit SUE formulation and the path-based algorithm, (*b*) a discussion of line search strategies, (*c*) a presentation of the numerical experiments, and (*d*) conclusions.

FORMULATION OF C-LOGIT SUE PROBLEM

To overcome the drawbacks of the MNL route choice model caused by its independence of irrelevant alternatives property, Cascetta et al. proposed the C-logit model to account for similarities between overlapping routes by adding a CF into the systematic term of the utility function, while keeping the analytical closed-form route choice probability expression (*12*). The C-logit model is stated as follows:

$$P_h^{rs}\left(\mathbf{c}^{rs}\right)=\frac{\exp\left(-\theta\left(c_h^{rs}+cf_h^{rs}\right)\right)}{\sum_{l\in H^{rs}}\exp\left(-\theta\left(c_l^{rs}+cf_l^{rs}\right)\right)}\qquad \forall\ h\in H^{rs}, r\in R, s\in S \tag{1}$$

where

R and S = sets of origins and destinations, respectively;
H^{rs} = set of all routes connecting the O-D pair (r, s);
P_h^{rs} = probability of choosing route h;
c_h^{rs} = deterministic (systematic) cost on route h, which is assumed to be the sum of all link costs on this route;
$\mathbf{c}$ = vector form of c_h^{rs};
cf_h^{rs} = CF of route h; and
θ = dispersion parameter that reflects the travelers' knowledge of network conditions.

The CF reduces the utility of overlapping routes and has the ability to alleviate the independence of irrelevant alternatives effect. Through the incorporation of the C-logit route choice model, more realistic SUE solutions can be generated than with the MNL model.

Cascetta et al. proposed several functional forms for the CF (*12*). The following form is considered in this paper:

$$cf_h^{rs}=\beta\ln\sum_{l\in H^{rs}}\left(\frac{L_{lh}}{\sqrt{L_h}\sqrt{L_l}}\right)^{\gamma}\qquad \forall\ h\in H^{rs}, r\in R, s\in S \tag{2}$$

where

L_{lh} = length of links common to routes l and h;
L_l and L_h = overall lengths of routes l and h, respectively; and
β and γ = two parameters.

If the parameter β is equal to zero, the C-logit model collapses to the MNL model. If β equals one, the C-logit choice probabilities in the limiting case of N coincident paths tend to $1/N$ of those calculated with a MNL model applied considering the coincident paths as a single path. The effect of CF in the route choice probability has been exemplified in simple networks by several authors [see Prashker and Bekhor (*10*) and Cascetta et al. (*12*), for example].

On the basis of the above discussion, the C-logit SUE traffic assignment problem can be formulated as the following variational inequality problem (*16*). That is, find $\mathbf{f}^*\in\Omega$, such that

$$\mathbf{F}\left(\mathbf{f}^*\right)^T\left(\mathbf{f}-\mathbf{f}^*\right)\geq 0\qquad \forall\mathbf{f}\in\Omega \tag{3}$$

where the mapping $\mathbf{F}$ is

$$\mathbf{F}\left(\mathbf{f}\right)=\mathbf{c}\left(\mathbf{f}\right)+\frac{1}{\theta}\left(1+\ln\mathbf{f}\right)+\mathbf{cf} \tag{4}$$

and the feasible set Ω consists of Equations 5–7:

$$q^{rs}=\sum_{h\in H^{rs}}f_h^{rs}\qquad \forall\ r\in R, s\in S \tag{5}$$

$$v_a = \sum_{r\in R}\sum_{s\in S}\sum_{h\in H^{rs}} f_h^{rs}\delta_{ha}^{rs} \qquad \forall\, a \in A \tag{6}$$

$$f_h^{rs} \geq 0 \qquad \forall\, h \in H^{rs}, r \in R, s \in S \tag{7}$$

where

f_h^{rs} = flow on route h, and $\mathbf{f}$ is its vector form,
c_h^{rs} = cost on route h, and $\mathbf{c}$ is its vector form,
$\mathbf{cf}$ = vector form of cf_h^{rs},
q^{rs} = travel demand between O-D pair (r, s),
T = transpose operation,
v_a = flow on $a \in A$ (A is the set of all links), and
δ_{ha}^{rs} = 1 if route h between O-D pair (r, s) uses link a, and 0 otherwise.

Equation 5 is the demand conservation constraint. Equation 6 is a definitional constraint that sums up all route flows that pass through a given link. Equation 7 is a nonnegativity constraint on the route flows.

In the above variational inequality formulation, the CF term could be either flow dependent [i.e., $\mathbf{cf}(\mathbf{f})$] or flow independent (i.e., $\mathbf{cf}$). These two cases correspond to the congestion-based C-logit (CC) and length-based C-logit (LC) SUE models. For the length-based model, the CF term is a constant (flow independent). Thus, this special case can be reformulated as an equivalent MP problem:

$$\min_{\mathbf{f}} Z(\mathbf{f}) = \sum_{a\in A}\int_0^{v_a} t_a(\omega)\,d\omega + \frac{1}{\theta}\sum_{r\in R}\sum_{s\in S}\sum_{h\in H^{rs}} f_h^{rs}\ln f_h^{rs} + \sum_{r\in R}\sum_{s\in S}\sum_{h\in H^{rs}} f_h^{rs} cf_h^{rs} \tag{8}$$

subject to

$$\mathbf{f} \in \Omega \tag{9}$$

where t_a is the travel time on $a \in A$. The above MP formulation can be regarded as an extension of Fisk's logit-based SUE formulation (*3*). Here, the third term is introduced in the objective function (Equation 8) to capture the route similarities. Note that $Z(\mathbf{f})$ is strictly convex with respect to f_h^{rs} and the constraints are all linear. Hence, the formulation has a unique route flow solution. For more detailed properties of the C-logit SUE problem, please refer to Zhou et al. (*16*).

GP ALGORITHM FOR C-LOGIT SUE PROBLEM

The route cost $\mathbf{F}(\mathbf{f})$ is generally nonadditive since it cannot be decomposed into a sum of link-based generalized costs on that route. Thus, the commonly used link-based traffic assignment algorithms (e.g., the Frank–Wolfe algorithm) are not applicable in this context. In addition, the ln$\mathbf{f}$ term explicitly includes route flow variables in the mapping. Hence, route-based algorithms are needed to solve the C-logit SUE problem.

The GP algorithm has been shown to be a successful route-based algorithm for solving the traditional traffic equilibrium problem, such as the standard user equilibrium problem, in Jayakrishnan et al. (*21*) and Chen et al. (*22*). As a result of its computational efficiency and simple implementation, a few software vendors (e.g., CUBE) are also beginning to provide the route-based GP algorithm to solve traffic assignment problems. A general iterative scheme of GP is as follows: given an initial solution $\mathbf{f}_0 \in \Omega$, the algorithm generates a sequence $\{\mathbf{f}_k\}$ according to the following recursive equation:

$$\mathbf{f}_{k+1} = P_\Omega\left[\mathbf{f}_k - \alpha_k \mathbf{F}(\mathbf{f}_k)\right] \qquad k = 0, 1, \ldots \tag{10}$$

where

$P_\Omega[\boldsymbol{\tau}]$ = unique projection of vector $\boldsymbol{\tau} \in R^{|H|}$ on Ω;
$|H|$ = cardinality of total route set $H = \cup H^{rs}$, $r \in R$, $s \in S$;
$\alpha_k \geq 0$ = judiciously chosen positive step size; and
k = iteration counter.

From the above description, the efficiency of the GP algorithm highly depends on two issues: (*a*) how to perform the unique projection operation on the constraint set, and (*b*) how to select a proper step size.

Projection Operation

With the projection operation, $P_\Omega[\tau]$, if Ω is a general polyhedron, then performing the projection operation is equivalent to solving a quadratic program problem; if Ω is a general convex set, then the projection operation is even more complicated. Thus, to take advantage of the GP, Ω should contain only simple constraints, such as nonnegativity constraints. Accordingly, the route flow vector $\mathbf{f}$ can be decomposed into two parts: the reference and nonreference routes, denoted by $\bar{\mathbf{f}}$ and $\tilde{\mathbf{f}}$, respectively. The reference route flows $\bar{\mathbf{f}}$ can be represented in terms of $\tilde{\mathbf{f}}$ according to the conservation constraint (Equation 5). In other words

$$\bar{f}_{\bar{h}_{rs}}^{rs} = q^{rs} - \sum_{h\in H^{rs},\, h\neq \bar{h}_{rs}} \tilde{f}_h^{rs} \qquad \forall r \in R, s \in S \tag{11}$$

where

$\bar{h}_{rs}$ = reference route between O-D pair (r, s),
$\tilde{f}_h^{rs}$ = flow on nonreference route h, and
$\bar{f}_{\bar{h}_{rs}}^{rs}$ = flow on reference route $\bar{h}_{rs}$.

Then the only solution variables are the nonreference route flows $\tilde{\mathbf{f}}$, and the new constraint set consists of only nonnegative constraints. Hence, the GP algorithm only requires a simple projection operation to be performed on the nonnegative orthant, without the need to solve expensive quadratic programs to ensure feasibility on the original polyhedron set (i.e., the conservation constraints).

Determination of Step Size

To guarantee the convergence of the GP algorithm, α_k must satisfy the following condition:

$$0 < \alpha_L \leq \alpha_k \leq \alpha_U < \frac{2\mu}{L^2} \tag{12}$$

where α_L and α_U are the lower and upper bounds of step size α and L is the Lipschitz constant of mapping $\mathbf{F}$, that is,

$$\|\mathbf{F}(\mathbf{f}) - \mathbf{F}(\mathbf{f}')\| \leq L\|\mathbf{f} - \mathbf{f}'\| \qquad \forall \mathbf{f}, \mathbf{f}' \in \Omega \tag{13}$$

and μ is the strongly monotone modulus of **F**, that is,

$$\left(\mathbf{f}-\mathbf{f}'\right)^{T}\left[\mathbf{F}\left(\mathbf{f}\right)-\mathbf{F}\left(\mathbf{f}'\right)\right]\geq\mu\left\|\mathbf{f}-\mathbf{f}'\right\|^{2}\qquad\forall\mathbf{f},\mathbf{f}'\in\Omega\tag{14}$$

The selection of α_k relies on the estimation of L and μ. However, L and μ are problem dependent. Thus, it is generally quite difficult to estimate these two constants in realistic problems, especially for the nonlinear mapping **F**. This is one of the research questions in this study.

In the following section, the detailed solution procedure of the general GP algorithm to solve the C-logit SUE problem is provided. A particular strategy for step size determination is not specified. Thus, different strategies can be embedded. To have a fair comparison of different step size strategies, a working route set was used. This set can be obtained from a route choice set generation algorithm (*23*). Behaviorally, it has the advantage of explicitly identifying those routes that are most likely to be used and also allows greater flexibility to include route-specific attributes that may not be obtainable directly from the link attributes (*24, 25*). A column generation procedure can also be readily embedded in the GP algorithm.

GP Algorithm

This section describes the GP algorithm.

Step 1. Initialization:

Set the tolerance error $\varepsilon > 0$ and the iteration counter $k = 0$.

Set the initial route flow vector $\mathbf{f}_0 \in \Omega$:

Calculate route costs $c_h^{rs} + cf_h^{rs}$ based on link free-flow travel times.

Calculate route choice probabilities P_h^{rs} according to Equation 1.

Calculate the initial route flows $\mathbf{f}_0$: $f_{h,0}^{rs} = q^{rs}P_h^{rs}$, $\forall h, r, s$.

Calculate the initial link flows: $v_{a,0} = \sum_{r\in R}\sum_{s\in S}\sum_{h\in H^{rs}} f_{h,0}^{rs}\,\delta_{ha}^{rs}$, $\forall a$.

Step 2. Update the link and route costs:

Update the link travel times: $t_a = t_a\,(v_{a,k})$, $\forall a$.

Compute the underlying mapping $\mathbf{F}_k$ according to Equation 4.

Step 3. Find the reference routes:

Find the reference route $\bar{h}_k^{rs}$ for each O-D pair based on $\mathbf{F}_k$: $\bar{h}_k^{rs} = \arg\min_{h\in H^{rs}}\left\{F_{h,k}^{rs}\right\}$.

Step 4. Route flow projection operation:

Update the nonreference route flows:

$$\tilde{f}_{h,k+1}^{rs}=\max\left[0,f_{h,k}^{rs}-\alpha_{k+1}\Gamma_{h,k}^{rs}\right]\qquad\forall\, h\in H^{rs},h\neq\bar{h}_k^{rs},r\in R,s\in S\tag{15}$$

where the step size α_{k+1} can be determined by different strategies, $\Gamma_{h,k}^{rs} = F_{h,k}^{rs} - F_{\bar{h}_k^{rs},k}^{rs}$ is the generalized travel cost difference between the nonreference route and the reference route, and set $f_{h,k+1}^{rs} = \tilde{f}_{h,k+1}^{rs}$, $\forall\, h\in H^{rs}, h\neq\bar{h}_k^{rs}, r\in R, s\in S$.

Update the reference route flows:

$$f_{\bar{h}_{h,k+1}^{rs}}^{rs}=q^{rs}-\sum_{h\in H^{rs},h\neq\bar{h}_k^{rs}}f_{h,k+1}^{rs}\qquad\forall r\in R,s\in S\tag{16}$$

Update the link flows:

$$v_{a,k+1}=\sum_{r\in R}\sum_{s\in S}\sum_{h\in H^{rs}}f_{h,k+1}^{rs}\delta_{ha}^{rs}\qquad\forall a$$

Step 5. Termination:

If the stopping criterion is met, stop; otherwise, set $k = k + 1$ and go to Step 2.

DETERMINATION OF STEP SIZE IN GP ALGORITHM

Several different strategies for step size determination have been suggested. In this section, three strategies for step size determination in the path-based GP algorithm are explored: (*a*) predetermined step size sequence, (*b*) Armijo step size, and (*c*) self-adaptive step size.

Predetermined Sequence of Step Size

Nagurney and Zhang showed that the projection method was convergent for strictly monotone variational inequality problems if the step size α_k were determined according to the following conditions (*26*):

$$\alpha_k>0$$

$$\lim_{k\to\infty}\alpha_k=0$$

and

$$\sum_{k=1}^{\infty}\alpha_k=\infty\tag{17}$$

The MSA, shown below, is a special case of the predetermined step size strategy.

$$\alpha_k=\frac{b_1}{b_2+k}\tag{18}$$

where $b_1 > 0$, $b_2 \geq 0$ are constants. In particular, $b_1 = 1$ and $b_2 = 0$ (i.e., $\alpha_k = 1/k$) correspond to the common version of the MSA. Although convergence is guaranteed using the MSA step size sequence, it suffers from the sublinear convergence rate. The convergence after the early iterations is quite slow.

Armijo Step Size

Bertsekas suggested a generalized Armijo rule to choose an appropriate step size, which could be regarded as a generalization of the well-known Armijo rule (*27, 28*). Given a nonstationary point $\mathbf{f}_k$, set

$$\alpha_k=\beta^{m_k}s\tag{19}$$

where m_k is the first nonnegative integer m that satisfies

$$Z\left(\mathbf{f}_k\right)-Z\left(\mathbf{f}_k\left(\beta^m s\right)\right)\geq\sigma\nabla Z\left(\mathbf{f}_k\right)^T\left[\mathbf{f}_k-\mathbf{f}_k\left(\beta^m s\right)\right]\tag{20}$$

where

$\sigma\in(0,1)$, $\beta\in(0,1)$, $s>0$ = fixed scalars,

Z = objective function, and

$$\mathbf{f}_k\left(\beta^m s\right)=P_x\left[\mathbf{f}_k-\beta^m s\nabla Z\left(\mathbf{f}_k\right)\right]\tag{21}$$

where **X** is the feasible set. This generalized Armijo rule is for constrained minimization problems, and the original Armijo rule is only for the steepest descent method in unconstrained minimization problems (*28*). If **X** is a nonnegative orthant and α_k is chosen according to Equations 19–21, it has been proved that without the requirement of the Lipschitz condition, every limit point of $\{\mathbf{f}_k\}$ generated by the GP algorithm is a solution point (*27*). The generalized Armijo rule belongs to the inexact line search strategies. At each step, the gradient information of the objective function and some objective function evaluations are required to determine an appropriate step size to improve the solution.

Self-Adaptive Step Size

The self-adaptive step size strategy was originally proposed by He et al. for the Goldstein–Levitin–Polyak projection algorithm (*29*). Recently, Chen et al. adopted this strategy in the GP algorithm to solve the nonadditive traffic equilibrium problem (*30*). The main idea of this strategy is to determine a suitable step size automatically from the information derived from previous iterations. This strategy is reminiscent of Bertsekas's generalized Armijo rule. However, it is more practical and robust since the step size sequence $\{\alpha_k\}$ is allowed to be nonmonotone (i.e., α_k can decrease as well as increase). Furthermore, the global convergence can be shown under certain assumptions on the underlying mapping **F** (*30*). Neither the Lipschitz constant nor the strongly monotone modulus are needed to design this strategy. The step sizes are automatically adjusted to satisfy the Lipschitz continuous condition and the strongly monotone assumption without a priori knowledge in the selection of the two constants. The detailed solution procedure is as follows:

Step 0. Parameter setting. Set step size $\alpha_0 > 0$, $\alpha_{\max} > 0$, $\gamma_0 = \alpha_0$ and parameters $\delta \in (0, 1)$, $u \in [0.5, 1]$.

Step 1. Self-adaptive scaling procedure: find the smallest nonnegative integer l_k

such that

$$\alpha_{k+1} = u^{l_k}\gamma_k$$

$$\tilde{f}^{rs}_{h,k+1} = \max\left[0, f^{rs}_{h,k} - \alpha_{k+1}\Gamma^{rs}_{h,k}\right] \qquad \forall\, h \in H^{rs}, h \neq \bar{h}^{rs}_k, r \in R, s \in S$$

$$\tilde{f}^{rs}_{h,k+1} = \max\left[0, f^{rs}_{h,k} - \alpha_{k}\Gamma^{rs}_{h,k}\right] \qquad \forall\, h \in H^{rs}, h \neq \bar{h}^{rs}_k, r \in R, s \in S$$

satisfy

$$(2-\delta)\alpha_{k+1}\left(\tilde{\mathbf{f}}_k - \tilde{\mathbf{f}}_{k+1}\right)^T\left(\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1}\right) - \alpha^2_{k+1}\left\|\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1}\right\|^2 \geq \max\left\{\frac{\alpha^2_{k+1} - \alpha^2_k}{\alpha^3_k}\left\|\tilde{\mathbf{f}}_k - \tilde{\mathbf{f}}_{k+1}\right\|^2, 0\right\} \tag{22}$$

where $\mathbf{\Gamma}_k$ is the vector form of $\mathbf{\Gamma}^{rs}_{h,k}$, and $\tilde{\mathbf{f}}_k$ and $\bar{\mathbf{f}}_k$ denote the vectors $\left(\ldots, \tilde{f}^{rs}_{h,k}, \ldots\right)^T$ and $\left(\ldots, \tilde{f}^{rs}_{h,k}, \ldots\right)^T$.

Step 2. Selection of γ_{k+1}.

If

$$0.5\alpha_{k+1}\left(\tilde{\mathbf{f}}_k - \tilde{\mathbf{f}}_{k+1}\right)^T\left(\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1}\right) - \alpha^2_{k+1}\left\|\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1}\right\|^2 \geq \max\left\{\frac{\alpha^2_{k+1} - \alpha^2_k}{\alpha^2_k}\left\|\tilde{\mathrm{f}}_k - \bar{\mathrm{f}}_{k+1}\right\|^2, 0\right\} \tag{23}$$

then

$$\gamma_{k+1} = \min\left\{\alpha_{k+1}/u, \alpha_{\max}\right\}$$

otherwise

$$\gamma_{k+1} = \alpha_{k+1}$$

Equation 22 is designed directly on the basis of the Lipschitz continuous condition (Equation 13, by $\|\mathbf{\Gamma}_{\tilde{k}} - \mathbf{\Gamma}_{k+1}\|$) and the strongly monotone condition [Equation 14, by $(\tilde{\mathbf{f}}_k - \tilde{\mathbf{f}}_{k+1})^T(\mathbf{\Gamma}_k - \mathbf{\Gamma}_{k+1})$]. The term $\|\tilde{\mathbf{f}}_k - \bar{\mathbf{f}}_{k+1}\|$ is the residual error $\|\tilde{\mathbf{f}}_k - P_{R+}[\tilde{\mathbf{f}}_k - \alpha_k\mathbf{\Gamma}_k]\|$. Hence, Equation 22 can be used to determine an appropriate step size that automatically satisfies these two conditions and further guarantees the convergence. For the detailed derivation, refer to Chen et al. (*30*).

The only difference between Equations 22 and 23 is $(2 - \delta)$ versus 0.5. Note that $0 < \delta < 1$ and $(2 - \delta) > 1$. If Equation 23 (with 0.5) is still satisfied, the current line search may be too conservative. The next initial step size can be enlarged by $\gamma_{k+1} = \min\{\alpha_{k+1} / u, \alpha_{\max}\}$; otherwise, the current acceptable step size α_{k+1} can be set as the next initial step size. Hence, Equation 23 can determine an appropriate initial step size for the next iteration, which is different from the fixed initial step size (s) for all iterations in the Armijo strategy. If s is set inappropriately, the acceptable step sizes may be far away from s. The search process from s to the acceptable step sizes may waste computational effort. The quality of s thus strongly affects the algorithmic performance. However, s is problem dependent, and it is nontrivial to choose an appropriate value without a priori knowledge. In contrast, the self-adaptive step size strategy adjusts the next initial step size (γ_{k+1}) and, consequently, the next acceptable step size according to the information derived from previous iterations (i.e., step size, route flows, and route costs). This treatment permits the initial and acceptable step size sequences to be nonmonotone. This mechanism makes the algorithm insensitive to the initial step size setting, thereby guaranteeing the robustness and efficiency of the algorithm. Thus, the self-adaptive step size strategy is capable of resolving the messy problem of finding an appropriate step size.

NUMERICAL EXPERIMENTS

This section describes two sets of numerical experiments conducted on the Winnipeg (Manitoba, Canada) network: (*a*) the computational performance of different line search strategies and (*b*) a comparison of the LC and CC SUE assignment results. The Winnipeg network consists of 154 zones, 1,067 nodes, 2,535 links, and 4,345 origin–destination (O-D) pairs. The network structure, O-D trip table, and link performance parameters are from the Emme/2 software (*31*). A working route set from Bekhor et al. is used (*25*). In this route set, there is a total of 174,491 routes, and the maximum number of routes for any O-D pair is 50.

The convergence criterion used in this paper is based on the root mean square error (RMSE) of route flows between two consecutive iterations:

$$\mathrm{RMSE} = \sqrt{\left\|\mathbf{f}_k - \mathbf{f}_{k-1}\right\|_2 / |\mathrm{H}|} \leq \varepsilon \tag{24}$$

where $\|\cdot\|_2$ is the Euclidean norm, and $|H|$ is the total number of routes. The dispersion parameter (θ) and the CF parameters (β and γ) are set at 1.2, 1.0, and 1.0, respectively. The initial step size of the MSA, Armijo, and self-adaptive strategies are all set at $\alpha_0 = 1.0$. The algorithms are coded in Compaq Visual FORTRAN 6.6 and run on a workstation with a Xeon 3.20-GHz processor and 2.00 GB of RAM.

Comparison of Line Search Strategies

The proposed self-adaptive step size strategy was compared with the widely used MSA and Armijo step size strategies. Their computational performances are shown in Figure 1. It can be observed that

- The MSA strategy has a fast convergence in the early iterations. However, it cannot achieve an accurate solution (e.g., 1 E−4) within an acceptable computational budget because of the sublinear convergence rate.
- The self-adaptive and Armijo step size strategies both have a linear convergence rate and can achieve an accurate solution. However, the self-adaptive strategy outperforms the Armijo strategy in terms of computational effort (2.2 h versus 9 h), especially in the latter iterations (or for a higher accuracy level).
- In the projection-type algorithms, the total number of projections is critical for computational efficiency. In terms of the number of outer iterations, the self-adaptive and Armijo strategies have a similar performance, as shown in Figure 1*b*. However, the self-adaptive strategy needs many fewer inner iterations (i.e., projections) per outer iteration. The average number of projections is 1.49 in the self-adaptive strategy, which is only about 30% of that in the Armijo strategy (i.e., 4.87). The reason is that the Armijo strategy always starts from a fixed initial step size (i.e., *s* in Equation 19). The acceptable step sizes of two consecutive iterations are usually very close, and they may be far away from *s* after the first several iterations. Thus, using a fixed initial step size may waste some computational efforts. In contrast, the initial step size per iteration

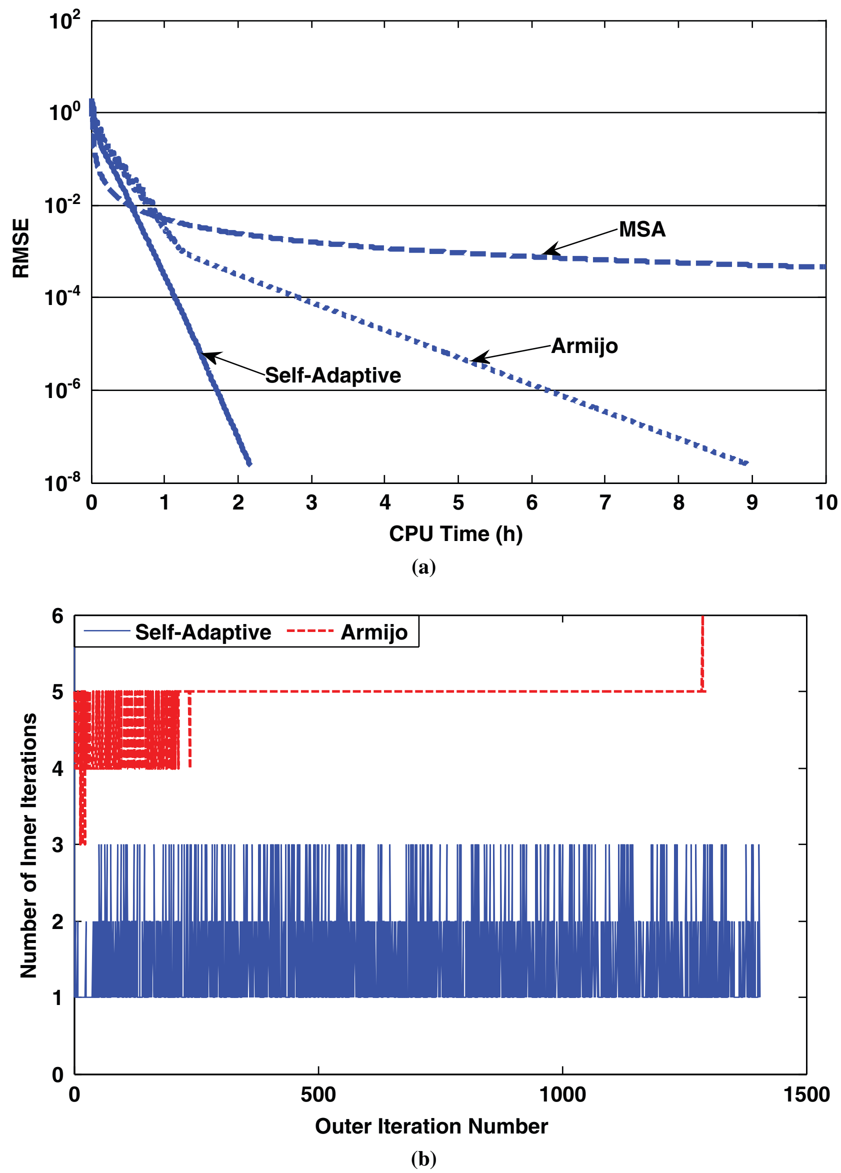

FIGURE 1 Computational performance of line search strategies: (*a*) convergence of three strategies for step size determination and (*b*) comparison of outer and inner iteration numbers.

TABLE 1 Computational Efforts Under Different Initial Step Sizes

Initial Step Size	0.001	0.01	0.1	1	10	Mean	SD	CV (%)
No. of iterations	870	890	913	933	957	913	34	3.76
No. of projections	1,320	1,352	1,397	1,413	1,468	1,390	57	4.10
CPU time (s)	4,832	4,947	5,061	5,180	5,287	5,062	181	3.57

NOTE: SD = standard deviation; CV = coefficient of variation; no. = number; CPU = central processing unit.

(except for the first iteration) in the self-adaptive strategy is smartly determined according to the previous iterative information as shown in Equation 23. This self-adaptive adjustment process makes the algorithm more efficient.

Sensitivity Analysis

The above analysis indicates that the proposed self-adaptive strategy performs better than other step size strategies in terms of computational efficiency. In the following analyses, the robustness and sensitivity of the GP algorithm embedded within the self-adaptive step size strategy is examined. Specifically, the effects of initial step size and dispersion and commonality parameters on the algorithmic computational performance are investigated. For demonstration purposes, the tolerance error (ε) is set at 1 E−5.

Initial Step Size

First, the impact of different initial step sizes α_0 on the computational performance of the algorithm is examined. Five initial step sizes with different orders of magnitude are shown in Table 1. It can be seen that the number of iterations, the number of projections, and the central processing unit time are all insensitive (or robust) to the initial step size value. The GP algorithm within the self-adaptive step size strategy is thus capable of reaching the convergent solution with a wide range of initial step size values while maintaining similar computational efforts.

Dispersion and Commonality Parameters

Second, the effects of dispersion (θ) and the CF (β) parameters on the algorithmic performance are examined. Nine combinations of θ and β values are shown in Figure 2. For a given value of β, the computational efforts significantly increase with the increase of θ. A large θ value means that the route travel time term plus the CF term [i.e., $\mathbf{c}(\mathbf{f}) + \mathbf{cf}(\mathbf{f})$] will dominate the mapping (or generalized route cost) $\mathbf{c}(\mathbf{f}) + \mathbf{cf}(\mathbf{f}) + (1 + \ln\mathbf{f}) / \theta$. The limiting case is the user equilibrium problem. The nonshortest route flows have a quite marginal impact on the convergence measure value, and the shortest route flow swap of two consecutive iterations could be quite large. Hence, more iterations and projections are needed to satisfy the convergence criterion for larger θ values. Larger β values tend to decrease the computational efforts. However, the effect is quite marginal. These results are similar to that in the cross-nested logit SUE model (*32*).

Special Case of LC SUE Model

If the CF (i.e., Equation 2) in the C-logit model is flow independent (only based on physical length), the CC model will degenerate to the LC model. In this case, the algorithm only needs to calculate the

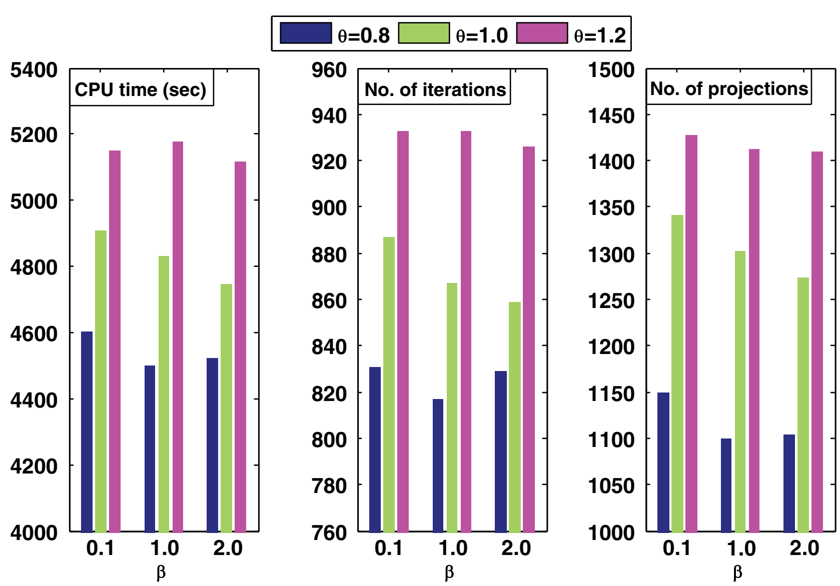

FIGURE 2 Computational efforts under different dispersion and commonality parameters.

TABLE 2 Route Choice Probabilities Under Different Models

Model	Variable	Route 1	2	3	4
MNL	Travel time **c**(**f**)	4.6000	5.3515	5.7666	5.4246
	Probability	0.4940	0.2005	0.1218	0.1836
LC	Travel time **c**(**f**)	4.6000	5.3515	5.7666	5.4247
	Commonality factor **cf**	0.5073	0.0000	0.7884	1.1441
	Travel cost **c**(**f**) + **cf**	5.1073	5.3515	6.5550	6.5688
	Probability	0.4773	0.3561	0.0840	0.0826
CC	Travel time **c**(**f**)	4.6000	5.3515	5.7666	5.4247
	Commonality factor **cf** (**f**)	0.5251	0.0000	0.8069	1.1726
	Travel cost **c**(**f**) + **cf** (**f**)	5.1251	5.3515	6.5735	6.5973
	Probability	0.4742	0.3614	0.0834	0.0811

NOTE: travel cost = travel time + commonality factor (**cf**). Route 1: link 219–link 1877. Route 2: link 218–link 1744–link 1747–link 1767. Route 3: link 220–link 1890–link 1885–link 1881. Route 4: link 220–link 1890–link 1884–link 1877. Different routes use the same links.

constant CF once for all iterations. Thus, the computational effort is significantly reduced (0.20 h for the LC model versus 1.44 h for the CC model). Depending on the modeling requirement for route overlapping, the computational effort could be quite different.

Route Choice Probability

The C-logit model was developed to resolve the independence of irrelevant alternatives issue of the MNL model. It introduces a CF to penalize the choice probability for overlapping routes. Now the CF's handling of the route overlapping and the difference between flow-dependent and flow-independent CFs is examined. Without loss of generality, only the results of four routes between O-D pair (106, 126) are shown.

From Table 2, it can be seen that all three models have almost the same route travel times, **c**(**f**). However, the models have quite different choice probabilities due to different considerations of route overlapping. On the one hand, both the LC and CC models have a CF for Routes 1, 3, and 4 (note that Route 2 has no coupling with other routes). This makes the deterministic disutilities in the two C-logit models (i.e., travel costs) significantly different from those in the MNL model (i.e., travel times). Note that Routes 1, 3, and 4 have overlaps with other routes. The consideration of the CF makes their deterministic disutilities larger and, consequently, decreases their choice probabilities. As a result of the probability conservation, the probability of Route 2 becomes much larger. Route 4 has the largest overlapping degree and, thus, has the largest probability reduction. On the other hand, the CC model explicitly considers the congestion effect on the CF. Thus the CF in the CC model is larger than in the LC model, further enlarging the deterministic disutilities. Again, the choice probabilities of Routes 1, 3, and 4 are reduced. To sum up, considering the route overlapping generally reduces the probabilities for routes coupled with other routes. Considering the congestion-dependent CF further penalizes these choice probabilities.

That the length-based and congestion-based models have different modeling specifications on route overlapping raises the question: What is the difference in their assignment results? In the following section, the assignment results of the two C-logit SUE models are compared. For comparison purposes, the classical MNL SUE model is also included as a benchmark.

Link Flow and Route Flow Patterns

The RMSE is used to quantify the equilibrium link–route flow difference between any two models. For example, the RMSE of link–route flows between the MNL and LC SUE models are

$$\text{RMSE}(\text{link flow}) = \sqrt{\frac{1}{|A|}\sum_{a\in A}\left(v_a^{\text{MNL}} - v_a^{\text{LC}}\right)^2}$$

$$\text{RMSE}(\text{route flow}) = \sqrt{\frac{1}{|H|}\sum_{r\in R}\sum_{s\in S}\sum_{h\in H^{rs}}\left(f_h^{rs,\text{MNL}} - f_h^{rs,\text{LC}}\right)^2} \quad (25)$$

where v_a^{MNL} ($f_h^{rs,\text{MNL}}$) and v_a^{LC} ($f_h^{rs,\text{LC}}$) are, respectively, the equilibrium link (route) flows from the MNL and LC SUE models. The nine combinations of β and θ values used in Figure 2 are also used in Figure 3. It can be seen that the effect of β on the equilibrium link and route flow patterns is significant. With the increase of β, the link flow and route flow differences are both significantly enlarged. The main reason is that parameter β can be regarded as the relative weight between the route travel time **c**(**f**) and the commonality factor **cf**. With the increase of β, **cf** gradually has more influence on the deterministic utility, which may affect the route choice probability more significantly. These results show that, depending on the model parameters, particularly β, the flow patterns of the LC and CC models may be quite different from the MNL model. Also, the CC model may be substantially different from the LC model. This further validates the importance of explicitly considering congestion-dependent route overlapping and also calibrating the commonality parameter accurately in route choice models.

CONCLUDING REMARKS

The C-logit model has been adopted as an extended logit model in many applications. In the route choice problem, it introduces a CF to the deterministic utility to resolve the route overlapping issue. In this paper, the C-logit SUE traffic assignment problem was investigated. Depending on the modeling specification for route overlapping, the CF could be flow dependent or flow independent, corresponding to the congestion-based and length-based models. This poses a challenge for algorithmic implementation.

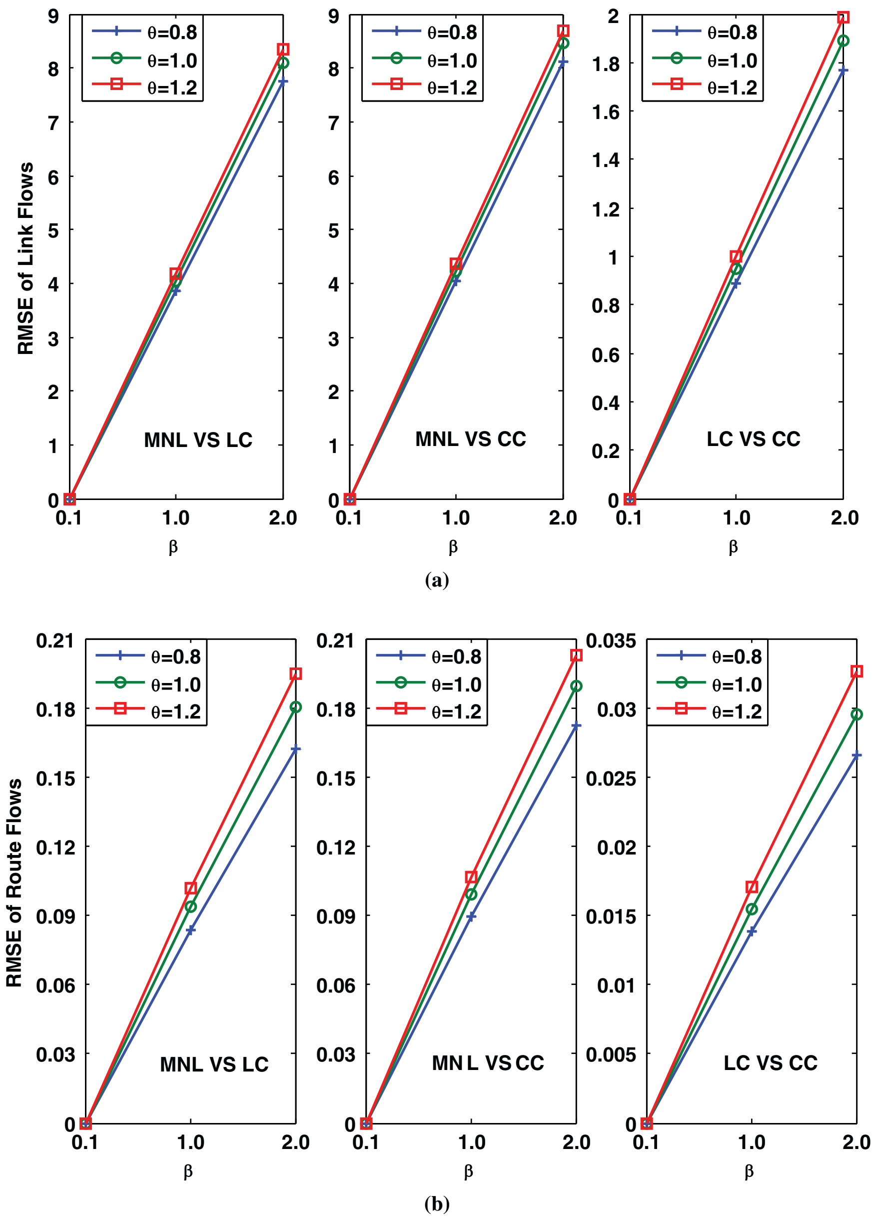

FIGURE 3 RMSE of various SUE models: (*a*) link flows and (*b*) route flows.

To solve the C-logit SUE models, new path-based algorithms were developed that were based on an adaptation of the GP algorithm. Three line search strategies (predetermined, Armijo, and self-adaptive) were examined. Two sets of experiments on the Winnipeg network were then conducted to examine the computational efforts of the different line search strategies and the impact of different modeling specifications for route overlapping.

The development and analysis of path-based algorithms performed in this paper represents a unique contribution, which could be used not only in the route choice problem but also for multidimensional travel demand problems, such as departure time, destination, mode, and route choices.

The analysis results revealed that the GP algorithm with the self-adaptive step size strategy performs better than other step size determination strategies. The MSA strategy has a fast convergence in the early iterations. However, it cannot achieve an accurate solution within an acceptable computational budget because of the sublinear convergence rate. The Armijo strategy is a widely used inexact line search strategy. However, it always starts from a fixed initial step size (s), which is nontrivial to choose without a priori knowledge. If s is set inappropriately, the acceptable step sizes may be far away from s. The search process from s to the acceptable step sizes may waste computational effort. The quality of s thus strongly affects the algorithmic performance. In contrast, the self-adaptive step size strategy adjusts the next initial and, consequently, the next acceptable step sizes according to the previous iterative information. This treatment permits the initial and acceptable step size sequences to be nonmonotone (i.e., to decrease as well as increase). This mechanism makes the algorithm insensitive to the initial step size setting, thereby guaranteeing the robustness and efficiency of the algorithm. Thus, the self-adaptive step size strategy is capable of resolving the messy problem of efficiently finding an appropriate step size. The contribution of the self-adaptive step size strategy to the GP algorithm mainly exhibits a significant reduction in the number of projections, which is quite beneficial for projection-type algorithms.

In addition, depending on the model parameters, particularly the CF parameter, the C-logit SUE flows might be quite different from the MNL SUE flows. The consideration of route overlapping generally reduced the probabilities for routes coupled with other routes. The consideration of congestion-dependent CF further penalized these choice probabilities. This consideration is quite important in the modeling of urban transport planning because of the dense road network (more overlapping) and high congestion levels. Neglecting or inappropriately modeling route overlapping might lead to biased route flow patterns. This biased route flow prediction may further affect the route-based network analysis (e.g., select link analysis, O-D trip table estimation, and traffic environmental impact analysis) based on route profiles of travel speeds. These results validated the need to explicitly model route overlapping and calibrate the CF parameter accurately in route choice models.

Some other path-based or origin-based algorithms were recently developed and then extended to efficiently solve other traffic assignment problems (e.g., the disaggregated simplicial decomposition algorithm, the origin-based algorithm, and projected gradient algorithm). To examine their application in solving the C-logit SUE problem, the self-adaptive step size strategy will be implemented in different algorithms, and their computational performance will then be compared.

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