Investigation of Stochastic Network Loading Procedures

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The network loading process of stochastic traffic assignment is investigated. A central issue in the assignment problem is the behavioral assumption governing route choice, which concerns the definition of available routes and the choice model. These two problems are addressed and reviewed. Although the multinomial logit model can be implemented efficiently in stochastic network loading algorithms, the model suffers from theoretical drawbacks, some of them arising from the independence of irrelevant alternatives property. As a result, the stochastic loading on routes that share common links is overloaded at the overlapping parts of the routes. Other logit-family models recently have been proposed to overcome some of the theoretical problems while maintaining the convenient analytical structure. Three such models are investigated: the C-logit model, which was specifically defined for route choice; and two general discrete-choice models, the cross-nested logit model and the paired combinatorial logit model. The two latter models are adapted to route choice, and simple network examples are presented to illustrate the performance of the models with respect to the overlapping problem. The results indicate that all three models perform better than does the multinomial logit model. The cross-nested logit model has an advantage over the two other generalized models because it enables performing stochastic loading without route enumeration. The integration of this model with the stochastic equilibrium problem is discussed, and a specific algorithm using the cross-nest logit model is presented for the stochastic loading phase.

Stochastic traffic assignment has received increased attention in recent years. A central issue in the assignment problem is the behavioral assumption concerning route choice. For a given transportation network, the stochastic loading is performed according to the route choice model. This report focuses on choice models, which are applied in stochastic loading procedures to solve the stochastic traffic assignment problem.

Traffic assignment models can be defined according to the behavioral model governing the choice among alternative routes. If we assume Wardrop's first principle, that travelers have perfect knowledge of travel times, the problem is known as deterministic traffic assignment. However, if we associate an error with the choice process, the behavioral model assumes that drivers minimize their perceived costs, and the problem is known as stochastic traffic assignment. The error association may have multiple sources, such as imperfect knowledge of travel times and measurement errors (such as the time-flow relationship); for route choice models, however, it generally is assumed that the error term is additive as in other discrete-choice models. In this report, the terms *time* and *cost* are used interchangeably, which means that travel costs are linear functions of travel times.

The stochastic loading procedure can be viewed as the "inner" problem of stochastic user equilibrium. The stochastic equilibrium is achieved by assuming a relationship between travel costs and congestion as in the deterministic case. The loading phase in stochastic assignment generally is more complicated and time intensive than that in the deterministic assignment. Instead of simply choosing the best route as in the deterministic assignment, stochastic loading models compute the probability of choosing each available route. In this case, an additional difficulty is to overcome path enumeration, and the solution is obtained either analytically or by simulation, depending on the route choice model. Therefore, in stochastic traffic assignment, an equilibrium solution is obtained at many more iterations than in the deterministic model.

The two main problems in route choice are the definition of the choice set (the available routes) and the choice model. In this study, how the different models handle these two problems was investigated. It is assumed that demand is constant over time, and network loading is performed by allocating to each available route, between an origin-destination (O-D) pair, the demand share according to the choice model.

Of the choice models used for stochastic assignment, the logit and probit are the most commonly used models. The multinomial logit (MNL) model can be efficiently implemented in stochastic assignment algorithms. The logit model, however, suffers from theoretical drawbacks, some of them arising from the independence of irrelevant alternatives (IIA) property. As a result, the stochastic loading on routes that share common links is overloaded at the overlapping parts of the routes. A more realistic model should take into account the similarity among routes that have many links in common, as in the "red bus-blue bus" problem.

Proposed is the adapting of existing discrete-choice models to route choice, in an attempt to overcome the overlapping problem. The behavioral model incorporates drivers' perception of available routes, apart from total route cost. All proposed models belong to the logit family, but with modified expressions that take into account the similarity among routes.

Recent developments in discrete-choice models, such as the C-logit of Cascetta et al. (*1*), specifically defined for route choice, were compared with two other generalized logit models: the cross-nested logit model of Vovsha (*2*) and the paired combinatorial logit model of Chu (*3*), both of which were adapted for route choice in this study.

STOCHASTIC LOADING PROCEDURES

First Models

Stochastic loading has been considered for more than 30 years. Von Falkenhausen (*4*) recognized the existence of nonattractive paths and proposed a heuristic procedure to reduce the number of paths between each O-D pair. The model suggested for route choice was the log-normal function. The probability of choosing a path was performed by evaluating the portion of the path that differed from the best path.

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Stochastic loading was performed by a Monte Carlo simulation at the path level. Burrell (*5*), who considered that link travel costs were random variables, distributed uniformly, performed stochastic loading by simulation at the link level.

Logit Model

Dial (*6*) proposed a very efficient procedure, based on the logit function, that obviated path enumeration. The probability of choosing path *k* is given by the logit formula as follows:

$$P(k) = \frac{\exp(-\theta c_k)}{\sum_l \exp(-\theta c_l)} \tag{1}$$

where c_k is the cost on path *k* and θ is a dispersion parameter.

It is possible to derive an expression for the probability of choosing path *k* at the link level as follows (*7*):

$$P(k) = \frac{\prod_{ij} W_{ij}}{\prod_j w_j} \tag{2}$$

where W_{ij} are the link weights and w_j are the node weights of path *k*, given as follows:

$$\begin{aligned} W_{ij} &= w_i * \exp\left[-\theta\left(c_i^* + c_{ij} - c_j^*\right)\right] \\ w_j &= \sum_i W_{ij} \end{aligned} \tag{3}$$

where c_{ij} is the cost of link (*i*,*j*), and c_i^*, c_j^* are the shortest distances to nodes *i*,*j*, respectively.

A very efficient algorithm using the above expressions was implemented, without explicitly having to store the available routes (the STOCH algorithm).

Regarding the choice set of available routes, Dial introduced the concept of an efficient path: For a given O-D pair, a path is reasonable only if it includes links that take the driver farther from the origin and closer to the destination. To implement an algorithm, the shortest path for each O-D pair must be computed. A much more efficient procedure can be implemented if the reasonable paths are defined only relative to one trip end (the origin or the destination). In this case, stochastic loading can be performed in a one-pass procedure that accounts for all destinations from a given origin (or destination) at once.

Dial's stochastic loading algorithm was used by many authors as the inner part of the stochastic user equilibrium (SUE) assignment because of its efficiency and also because the logit model has an equivalent minimization problem.

Dial's procedure suffers from theoretical problems, some of them arising from the MNL function with linear costs (which accounts for the difference between path costs, regardless of absolute values) and some of them arising from the IIA property (because the flow in Dial's algorithm is distributed equally for equal-cost paths, regardless of network topology). Therefore, in a network in which some routes share many common links, the overlapping effect will result in unrealistic excess flows in these links with respect to disjointed routes (no links in common). Daganzo and Sheffi (*8*) presented examples of these problems.

Methods to overcome some of the problems arising from Dial's method have long been investigated. Gunarsson (*9*) and Tobin (*10*) developed variants of the STOCH algorithm, but with non-logit-based functions for route choice. Bell (*11*) presented a logit-based algorithm in which the efficient path definition was relaxed to include all routes, including cycles (a link can be part of a path many times). Randle (*12*) presented a model in which the choice among routes was based on the exclusive links of the routes. His idea was to check the second-best path to each node in the network from a given origin. The loading procedure is then performed by tracing the path backwards, until there is a common node (not second best). The probability of using each subroute is computed by assuming a uniform distribution of link costs. The advantage of this method is that no explicit route enumeration is needed, and the loading process takes (partially) into account the effect of overlapping routes.

Probit Model

The multinomial probit (MNP) model is another discrete-choice model used to tackle problems arising from the logit model. The model, first suggested for route choice by Daganzo and Sheffi (*8*), can cope with the overlapping problem, but a solution cannot be obtained in closed form. The stochastic loading algorithm proposed was similar to Burrell's simulation method.

Maher (*13*) presented a stochastic assignment method based on Clark's analytical approximation. He proposed a stochastic loading algorithm without the need for route enumeration. The method progresses through the network by using a Markov assumption, which means that the pattern of route choice from the origin to an intermediate node is independent of the pattern of route choice from this node to the destination. In a recent study, Maher and Hughes (*14*) refined the model to solve the stochastic loading for more general networks and to include elastic demand. Clark's approximation, however, is not suitable when the number of alternatives becomes high, and the solution for real-size networks can be performed only by simulation methods.

Yai et. al (*15*) recently proposed the MNP model with a structured covariance matrix to represent the overlapping between route alternatives. The explicit evaluation of the covariance matrix is needed to compute the choice probabilities. Their idea was to relate the covariance matrix terms to measurable overlapping variables; for example, the covariance between two roads is proportional to the common length of the routes. The stochastic loading then is performed by numerical integration.

All stochastic models presented assume that the random sources are additive. For example, the assumption of drivers' imperfect knowledge of travel costs can be viewed as if there are "true" travel times on the network and drivers choose their routes based on the expected travel times. Mirchandani and Soroush (*16*) presented a route choice model based on an additional source of stochasticity. In their model, the network itself is stochastic, and the assumption was that travelers associate the stochasticity of a path with the risk of choosing that path. Tatineni et al. (*17*) presented experiments that assumed exponential functions for the risk-taking behavior.

Explicit Route Choice Set

Stochastic loading with explicit route enumeration generally is performed by some heuristic rule to restrict the number of available routes. Ben-Akiva et al. (*18*) proposed a labeling technique that enumerates paths according to some exclusive aspect (label) or combination of different labels (shortest path, cheapest path, maximized motorway links, and so on). The nested logit was proposed to model the choice process, which was performed in two stages: First, a choice set of labeled routes is generated; second, the probability of choosing one of these routes is computed.

The logit model also was implemented with explicit route methods. Leurent (*19*) extended the STOCH algorithm to reduce the number of routes between an origin-destination (O-D) pair. The idea is to compute the routes that are not too long compared with the shortest path. De La Barra et al. (*20*) presented a heuristic method to compute the *k*-shortest routes, in which the next shortest route is found after penalizing the links that compose the previously shortest route.

When explicit paths are available, more sophisticated choice models can be used. Cascetta et al. (*1*) proposed a new model, C-logit, which accounts for the overlapping part of different paths. The formulation is as follows:

$$P(k) = \frac{e^{c_k - cf_k}}{\sum_l e^{c_l - cf_l}} \tag{4}$$

where cf_k, the commonality factor of path k, is a measure of the degree of similarity of path k with other paths in an O-D pair.

The commonality factor can be specified in different ways; the formula adopted by Cascetta et al. (*21*) is as follows:

$$cf_k = \beta \ln \sum_l \left(\frac{L_{kl}}{L_k^{1/2} L_l^{1/2}} \right)^{\gamma} \tag{5}$$

where L_{kl} is the length (cost) of links common to paths k and l, and L_k, L_l are the overall path lengths (costs) of paths k and l, respectively.

The C-logit model can take into account the similarity between each pair of routes independently. In the nested logit model, for example, all routes in a common grouping are required to be similar.

NEW STOCHASTIC LOADING PROCEDURES

An ideal procedure for stochastic loading should have the following properties: It takes into account the overlapping between different paths, the model is analytically tractable, the choice set is not restrained by heuristic techniques, and the algorithm can be implemented with reasonable computer resources. This section deals with the first two properties.

The logit family of models is analytically appealing, and the challenge was to overcome the overlapping problem. Cascetta et al. (*1*) were the first to introduce a modification to the MNL model to solve this problem. This study follows their approach, in which two discrete-choice models are adapted to cope with the overlapping problem, and convenient analytical expressions are maintained. The proposed models have been studied by others (*2*,*3*), and their application to route choice models is defined and investigated here.

Cross-Nested Logit Model

The cross-nested logit model, developed by Vovsha (*2*), was applied for a mode-choice situation. The model was defined as a particular case of the generalized extreme value (GEV) case as follows:

$$G(y_1, y_2, \ldots, y_n) = \sum_m \left(\sum_k \alpha_{mk} y_k \right)^{\mu} \tag{6}$$

where $y_k = \exp(V_k)$.

This function satisfies conditions for serving as a basis of the distribution of random utilities (*22*). The probability of choosing an alternative (route) k then is obtained as follows:

$$P(k) = \frac{e^{V_k + \ln \sum_m \alpha_{mk} \left(\sum_l \alpha_{ml} e^{V_l} \right)^{\mu - 1}}}{\sum_j e^{V_j + \ln \sum_m \alpha_{mj} \left(\sum_l \alpha_{ml} e^{V_l} \right)^{\mu - 1}}} \tag{7}$$

where the utility V_k is assumed to be a linear combination of the path cost c_k.

The formulation of the cross-nested model presented above permits an alternative (in this case, a route) to belong to more than one nest (in this case, a link). The crossing effect is represented by the inclusion coefficient α_{km}, $0 \le \alpha_{km} \le 1$. The nested logit model is a special case of the cross-nested model, in which α_{km} is equal to 0 (the alternative is the only one in the nest) or to 1 (the alternative belongs to a specified nest).

It is possible to rewrite the expression for the choice probability as follows:

$$P(k) = \sum_m P(m) P(k|m) \tag{8}$$

where the conditional probability of a route k being chosen in link (nest) m is

$$P(k|m) = \frac{\left(\alpha_{km} e^{-c_k} \right)^{1/\mu}}{\sum_l \left(\alpha_{lm} e^{-c_l} \right)^{1/\mu}} \tag{9}$$

and the marginal probability of a nest m being chosen is

$$P(m) = \frac{\left[\sum_k \left(\alpha_{km} e^{-c_k} \right)^{1/\mu} \right]^{\mu}}{\sum_b \left[\sum_k \left(\alpha_{kb} e^{-c_k} \right)^{1/\mu} \right]^{\mu}} \tag{10}$$

The probability of choosing route k depends on two factors: the generalized cost of the route c_k, and the inclusion coefficient α_{km} in the links m that form the route k. It is possible to define a functional relationship for the inclusion coefficient with respect to the links in a route. For example, a natural measure can be specified as follows:

$$\alpha_{km} = \frac{L_m}{L_k} \delta_{mk} \tag{11}$$

where L_m is the link length and L_k is the path length.

In this case, the inclusion coefficient is dependent only on network topology. If we assume that that the inclusion coefficient is proportional to the link costs (instead of link lengths), then α_{km} also is dependent on congestion.

The coefficient μ indicates the degree of nesting. When μ is equal to 1, the model is equal to the multinomial logit. As the degree of nesting becomes higher, $\mu \to 0$, the model becomes probabilistic at the higher (link) level and deterministic at the lower (nest) level. This extreme case is suitable for route choice, because completely overlapping paths will be perceived as a single path and mutually exclusive paths will be distributed only according to path costs (as with the multinomial logit).

According to Vovsha (*2*), the marginal probability of choosing a nest reduces to

$$\lim_{\mu \to 0} P(m) = \frac{e^{\ln \alpha_{mk(m)} - c_{r(m)}}}{\sum_b e^{\ln \alpha_{bk(b)} - c_{r(b)}}} \tag{12}$$

where $k(m)$ is the best route of an O-D pair passing through link m (shortest m-path). The conditional probability of choosing route k in a given link m is simply

$$\lim_{\mu \to 0} P(k|m) = \begin{cases} \frac{1}{|R(m)|}, & k \in R(m) \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

where $|R(m)|$ is the number of equal shortest m-paths.

Because the probability of a tie between shortest m-paths is low for realistic networks, the conditional probability can be omitted, and thus the probability of choosing a route equals the marginal probability of choosing a nest.

Paired Combinatorial Logit Model

Another GEV-type model, proposed by Chu (*3*) and later developed by Koppelman and Wen (*23*), is the paired combinatorial logit (PCL) model. It also can be applied to model route choice. The G function in this case is as follows:

$$G(y_1, y_2, \ldots, y_n) = \sum_{k=1}^{n} \sum_{j=k+1}^{n-1} (1-\sigma_{kj}) \left(y_k^{\frac{1}{1-\sigma_{kj}}} + y_j^{\frac{1}{1-\sigma_{kj}}} \right)^{1-\sigma_{kj}} \tag{14}$$

The probability of choosing an alternative (route) k is given as follows:

$$P(k) = \frac{\sum_{j \neq k} e^{\frac{V_k}{1-\sigma_{kj}}} (1-\sigma_{kj}) \left(e^{\frac{V_k}{1-\sigma_{kj}}} + e^{\frac{V_k}{1-\sigma_{kj}}} \right)^{-\sigma_{kj}}}{\sum_{l=1}^{n-1} \sum_{m=l+1}^{n} (1-\sigma_{lm}) \left(e^{\frac{V_l}{1-\sigma_{lm}}} + e^{\frac{V_m}{1-\sigma_{lm}}} \right)^{1-\sigma_{lm}}} \tag{15}$$

where σ_{kj} is an index of similarity between alternatives k and j.

The double summation includes $n(n-1)/2$ elements, which is the number of different pairs of alternatives in the choice set of n alternatives. If σ_{kj} is equal to 0 for all k, j pairs, the PCL collapses to the MNL model. The PCL model allows a differential correlation between pairs of alternatives, as can be seen from the following development:

$$P(k) = \sum_{k \neq j} P(kj) P(k|kj) \tag{16}$$

where $P(k|kj)$ is the conditional probability of choosing alternative k, given the chosen binary pair k, j as follows:

$$P(k|kj) = \frac{e^{\frac{V_k}{1-\sigma_{kj}}}}{e^{\frac{V_k}{1-\sigma_{kj}}} + e^{\frac{V_j}{1-\sigma_{kj}}}} \tag{17}$$

and $P(kj)$ is the marginal probability for the binary pair k,j as follows:

$$P(kj) = \frac{(1-\sigma_{kj}) \left(e^{\frac{V_k}{1-\sigma_{kj}}} + e^{\frac{V_k}{1-\sigma_{kj}}} \right)^{1-\sigma_{kj}}}{\sum_{l=1}^{n-1} \sum_{m=l+1}^{n} (1-\sigma_{lm}) \left(e^{\frac{V_l}{1-\sigma_{lm}}} + e^{\frac{V_m}{1-\sigma_{lm}}} \right)^{1-\sigma_{lm}}} \tag{18}$$

In the well-known nested logit model, all pairs of alternatives in a common grouping are required to be similar. In the PCL model, each pair of alternatives can have a similarity relationship that is completely independent of the similarity relationship of other pairs of alternatives. This feature is highly desirable for route choice models, because each pair of routes may have different similarities.

Like the cross-nested logit model, which was adapted for route choice by defining the inclusion coefficient, it is possible to relate the similarity index to the network topology. The functional form is similar to the C-logit model as follows:

$$\sigma_{kj} = \frac{L_{kj}}{(L_k L_j)^{0.5}} \tag{19}$$

where L_{kj} is the length (cost) of the common part of routes k and j.

Equation 19 confines the similarity-index boundaries between 0 and 1. These conditions have to hold for the PCL model to be consistent with random utility maximization. If σ_{kj} approaches 1, this indicates that all the links of a path are completely equal to the links of the other path (maximum overlap). However, if the similarity index is 0, this means that the paths have no link in common (disjointed paths).

NUMERICAL EXAMPLES

In this section, the logit-family of models (MNL, C-logit, cross-nested logit, and PCL) are compared for simple networks. Concentration is on the different logit formulations, and the MNL serves as a basis for comparison. In all examples, the utility function is considered to be linearly dependent on the link travel costs, and the dispersion parameter θ is constant and equal to 1. For the C-logit model, the commonality factor is defined by Equation 5, with parameters β and γ equal to 1. The inclusion coefficient in the cross-nested logit model is defined by Equation 11 and the similarity index in the PCL model by Equation 19.

Example 1: Two Correlated Routes

The example of two correlated routes has been used by many authors to illustrate the insensitivity of the MNL to network topology. Consider the loophole problem, with three available routes between a single O-D pair, as shown in Figure 1. The travel costs on each link are shown in the figure. This example was used by Cascetta et al. (*1*) to compare the C-logit and MNL.

In this example, it is assumed that the overall path costs are equal to b, and it is sufficient to vary the cost on link a to show how the different models consider the overlapping effect.

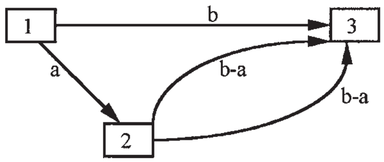

FIGURE 1 Two correlated routes (Example 1).

Figure 2 shows the probability of choosing route 1–3 (the upper route) for different link costs on link *a*. The overall cost was set to 10 units (*b* = *10*); as the cost on link *a* increases, the probability of choosing route 1 increases, because the two overlapping routes passing through node 2 become more similar to each other.

As explained above, the MNL depends only on the total route cost. Therefore, this model results in an equal probability for each of the routes, regardless of the degree of overlapping. It is interesting that both PCL and C-logit give results that are very close to each other; however, this phenomenon is particular to the simple case, because the similarity index in the PCL model is $\sigma_{12} = \sigma_{13} = 0$, $\sigma_{23} = \sigma = a/b$, and the commonality factor in the C-logit model reduces to $cf_1 = 0$, $cf_2 = cf_3 = cf = \ln(1 + a/b)$. The probability of choosing the upper route in the C-logit model is as follows:

$$P(1) = \frac{e^{-b}}{e^{-b}\left[1 + \frac{2}{1+\frac{a}{b}}\right]} = \frac{1+\frac{a}{b}}{3+\frac{a}{b}} \qquad (20)$$

and the probability of choosing the upper route in the PCL model is as follows:

$$P(1) = \frac{2e^{-b}}{4e^{-b} + (1-\sigma)2^{1-\sigma}e^{-b}} = \frac{1}{2+(1-\sigma)2^{-\sigma}} \qquad (21)$$

To obtain exactly the same probability for the C-logit and PCL models, the following relationship between the commonality factor and the similarity index can be obtained analytically:

$$cf = \ln\left[\frac{2}{(1-\sigma)2^{-\sigma}+1}\right] \qquad (22)$$

The result for the cross-nested model for different nesting coefficients also is shown in Figure 2. It can be seen that when $\mu \to 0$, the model is more sensitive to the overlapping effect. As previously shown, the extreme degree of nesting in the cross-nested logit is suitable for route choice. When $\mu \to 1$, the model becomes closer to MNL, as expected, and therefore the intermediate cases $0 < \mu < 1$ will not be considered in the examples that follow.

Example 2: Three Correlated Routes

In the example of three correlated routes, there are three available routes from 1 to 4: the upper route 1-2-4, the lower route 1-3-4, and the *Z*-shaped route 1-2-3-4, as shown in Figure 3.

In all three available routes from 1 to 4, the overall path cost is equal to $a + b$. In this network, the upper and lower routes overlap with the *Z* route by the same amount *a*, although in different links. Figure 4 shows the probability of choosing the *Z* route for different values of a/b. The cost on link *b* was kept constant and was equal to 10 units.

As a/b increases, the network changes from being a straight three-link choice (with equal probabilities in each route) to being close to a figure-eight network, in which the crossing can be made only from the upper to the lower route. As in the previous example, the MNL model is not sensitive to the overlapping effect. All other models

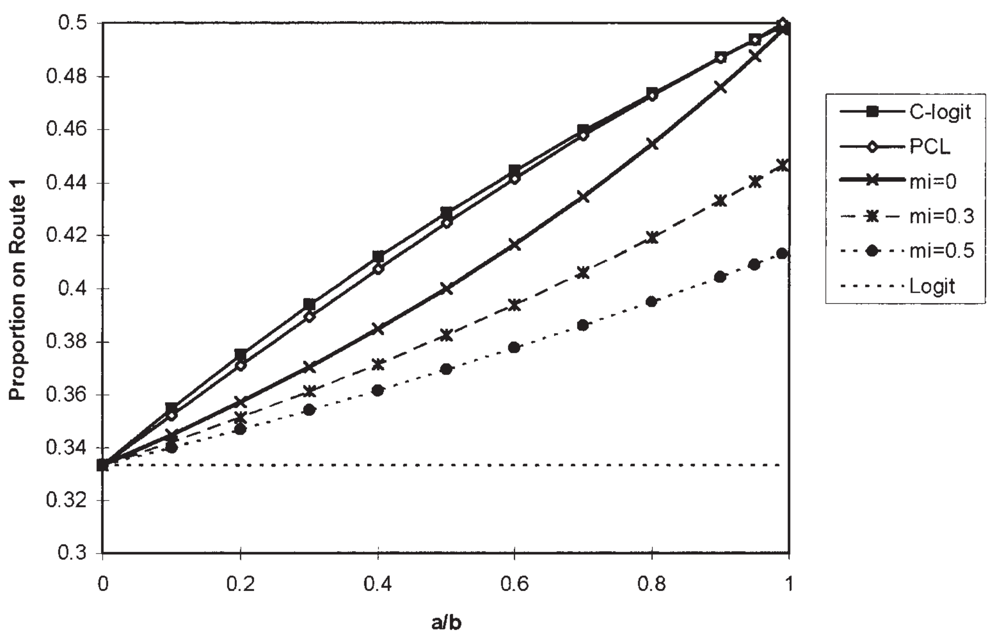

FIGURE 2 Comparison of logit-type models (Example 1).

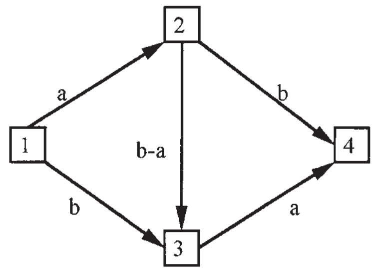

FIGURE 3 Three correlated routes (Example 2).

take into account the increasing similarity of the Z route with the two other routes, as a/b increases. As a consequence, the probability of choosing the third route decreases, as shown in the figure, but with different slopes for the different models.

In this example, the PCL model is the most sensitive to the overlapping effect; next comes the cross-nested logit model. The C-logit model becomes less sensitive than the other models, as a/b increases. This phenomenon happens with C-logit because the commonality factor in this example is equal to the following:

$$cf1 = cf2 = \ln\left(1 + \frac{a}{a+b}\right)$$
$$cf3 = \ln\left(1 + \frac{2a}{a+b}\right) \tag{23}$$

the probability of choosing the third route is easily obtained:

$$P(3) = \frac{2a+b}{8a+3b} \tag{24}$$

Example 3: Short Bypass–Long Bypass

In the example of short bypass–long bypass, one route is shorter than the two other equal-cost routes. This problem deals with the assumption of linearity between the utility function and travel costs. Figure 5 represents the network.

The travel cost on the straight route from 1 to 4 is equal to $2a + b - 1$, and the travel costs on routes 1-2-4 and routes 1-3-4 are equal to $2a + b$. Again, the performances of the different models are compared for increasing values of the ratio a/b, with the total route costs kept constant and equal to 20 (the shortest straight route) and 21 (the two other routes). In this case, this ratio can vary from 0 (three disjointed routes) to a very large number (very similar routes). The probability of choosing the straight route for $a = 0$ is equal (for all models) to $1/(1 + 2/e) = 0.576$. Figure 6 shows the results.

It is expected that in the case of common overlapping links between routes (as considered in this example), the choice is based on the differences between the nonoverlapping links. Sheffi (*24*) also considered this problem to show the advantages of the probit model over the logit model. Because linear travel costs are assumed for the utility functions, both the MNL and the C-logit models will give the same response for any degree of overlapping, as shown in Figure 6. This happens because the models are sensitive only to the absolute difference between the links. The PCL and the cross-nested logit models take into account the relative difference between the links, as does the probit model.

In the cross-nested logit model, the inclusion coefficient α, defined as the ratio between link and path costs, is very sensitive to the degree of overlapping. The PCL model gives intermediate results, because the similarity index α for each of the routes tends to be equal for increasing values of a/b; thus for high overlapping links, the probability of choosing the straight route stabilizes.

EQUILIBRIUM CONSIDERATIONS

As indicated above, stochastic loading is part of the stochastic user equilibrium problem. The first consideration is related to the route enumeration problem. The logit-type models presented can solve the overlapping problem at the expense of storing the alternative routes. To implement such models for real-sized networks, two main approaches can be taken in regard to the explicit route enumeration.

The first approach is to confine the large number of available routes to a fixed (small) amount before the assignment procedure,

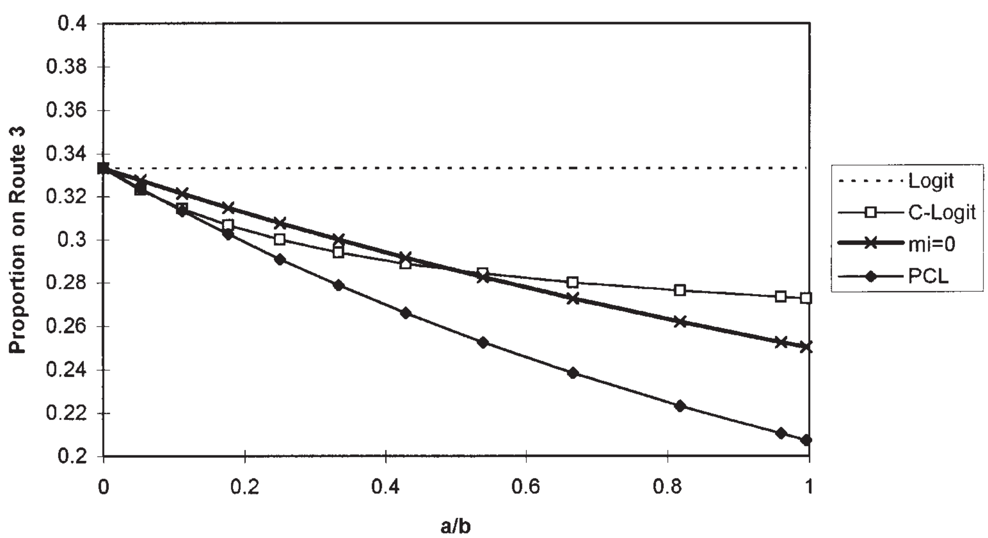

FIGURE 4 Comparison of logit-type models (Example 2).

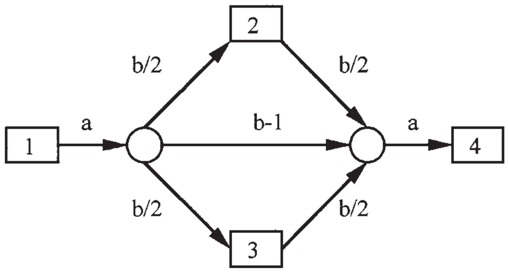

FIGURE 5 Short bypass–long bypass (Example 3).

and to perform the stochastic loading for the same set of routes at each iteration. Cascetta et al. (*21*) addressed this problem and devised a heuristic method to confine the available routes to a reasonable number. Similarly, the PCL model also can be applied by using the same approach.

A different algorithmic approach is to generate routes as part of the iterative process in the assignment. The idea is to perform a stochastic loading with the routes currently available and to achieve convergence by exploring both the congestion effect and the stochastic loading effect. In this case, it is possible to use the logit model, employing Fisk's (*25*) optimization problem to solve the SUE problem as follows:

$$\text{Min} \int_0^{x_a} t_a(w)dw + \frac{1}{\theta}\sum_{rs}\sum_{k} f_k^{rs} \ln f_k^{rs}$$

$$\text{s.t.} \sum_k f_k^{rs} = q^{rs}, \quad \forall\ r,s$$

$$f_k^{rs} \geq 0, \qquad \forall\ k,r,s \tag{25}$$

where f_k^{rs} is the flow on route k for O-D rs, q^{rs} is the demand between r and s and t_a is the travel time (cost) on link a, which is a function of the link flow x_a.

The solution for this problem is obtained by the logit model. Chen and Alfa (*26*), Bell et al. (*27*), Leurent (*19*), Akamatsu (*28*), and Damberg et al. (*29*) presented algorithms to solve the SUE problem in evaluating Fisk's function.

The solution of the SUE problem with models other than logit can be achieved with the well-known method of successive averages algorithm of Sheffi and Powell (*30*). The main advantage of this algorithm is that stochastic loading can be performed at the link level; thus, implicit methods are very suitable. The most common method is to simulate travel costs at the link level according to some distribution (e.g., normal), and then perform a stochastic loading by averaging successive trials. To obtain route flows according to the choice model, however, many sampling iterations must be performed, because the probit model has no known equivalent minimization program as the logit model has.

Analytical methods such as Dial's can perform implicit path enumeration (STOCH algorithm). The logit model can be efficiently implemented in such an algorithm, but the logit model suffers from theoretical drawbacks, as previously explained. The cross-nested logit model, however, also may be implemented in an algorithm without route enumeration. This is possible because of the combination of two elements: the inclusion coefficient, defined as the ratio between the link (nest) and path costs, and the nesting coefficient $\mu \rightarrow 0$. Because the nesting is defined at the link level, it is sufficient to check the inclusion coefficient for each link in the network (for each O-D), and then the marginal probability of choosing a nest (link) may be evaluated with Equation 11. The implementation of the cross-nested logit model in a stochastic network loading algorithm without route enumeration is shown in the next section.

The cross-nested logit, and perhaps other GEV-type models, may provide a solution for the equivalent minimization problem, just as the MNL model is the solution for Fisk's problem. The idea is adapted from Fernandez et al. (*31*), who developed different models to solve network equilibrium models with combined modes. The formulation of an equivalent minimization problem is obtained by adding an entropy term specific to each mode leg (in their case, the combined mode was auto and metro).

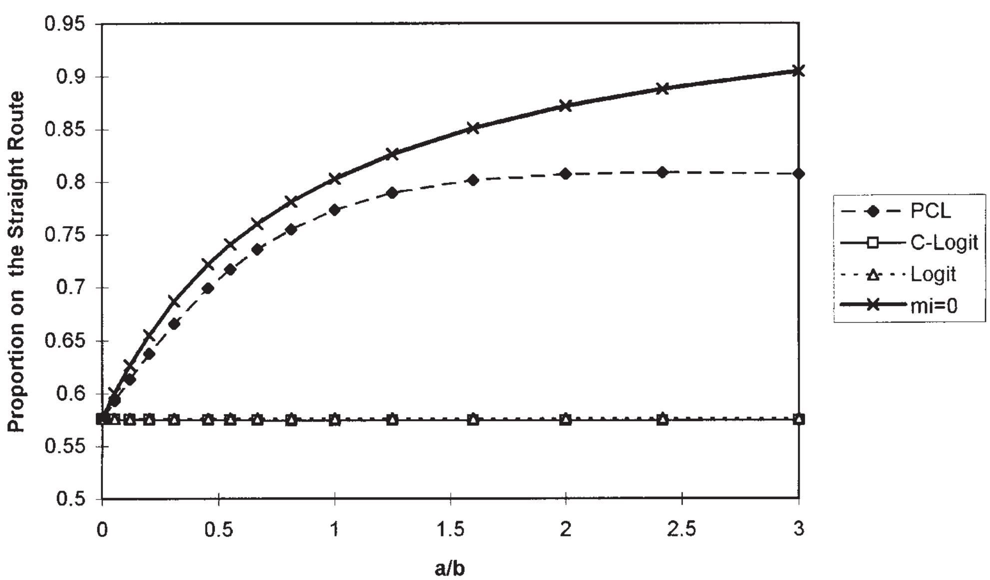

FIGURE 6 Comparison of logit-type models (Example 3).

Fisk's minimization problem has one term corresponding to the congestion part and another term to the stochastic effect (the entropy term). The cross-nested logit model has two levels of stochasticity: the link level (the marginal probability) and the route level (the conditional probability). In this way, it is possible to add another entropy term to the original objective function that corresponds to the marginal probability of choosing a link (nest).

CROSS-NEST LOGIT LOADING ALGORITHM

The cross-nest logit loading algorithm falls into the category of implicit path enumeration, because an a priori enumeration of the paths is not needed. The key problem is to compute the marginal probability for choosing a link (Equation 12). To do so, the shortest path passing through each link *m* (shortest *m*-path) must be computed for each O-D pair. This computation poses two extra difficulties, compared to standard shortest path procedures: (*a*) the shortest *m*-path must be found for each O-D separately; and (*b*) the shortest *m*-path is a bicriterion problem, because the objective function is to minimize the logarithm of the inclusion coefficient α with the cost function.

To alleviate the nonadditive logarithm term, α is assumed to be a function of the route cost, so that minimizing the whole function can be simplified to minimizing the additive cost function. Because the problem is to find the shortest path passing through each link, use is made of the cascade-matrix algorithm of Floyd (*32*). The shortest *m*-path in this case can be traced by the shortest path from the origin to each link *m,* and from link *m* to the destination.

Once the shortest *m*-path is found, the marginal probabilities can be computed easily. The loading process is then performed for all links at once, because in accordance with Equation 12, the denominator is simply the sum of each shortest *m*-cost for each nest (link) *m.*

The stochastic loading algorithm may be summarized as follows:

1. Find the shortest path between all nodes in the network.
2. Store the distance matrix *D* and predecessor matrix *P.*
3. For each origin-destination pair $k = i,j$.
4. For each nest (link) *m.*
5. $c_{k(m)} = D[i,m] + D[m,j]$.
6. Trace back the shortest *m*-path with matrix *P*; load each link *b* onto the *m*-path with

$$\text{load}\ (b) = \text{load}\ (b) + \exp(\ln \alpha_{km} - c_{k(m)})$$

7. $\text{total\_load} = \text{total\_load} + \exp(\ln \alpha_{km} - c_{k(m)})$
8. For each link *m,* $\text{flow}\ (m) = \dfrac{\text{load}\ (m)}{\text{total\_load}}$
9. Next *m.*
10. Next *k.*

This algorithm can be improved by implementing more efficient shortest-path procedures. Another improvement is to relax the O-D constraint to achieve the loading in a one-pass fashion.

CONCLUSIONS

The stochastic network loading process was investigated. The MNL model is used as the inner part of the stochastic user equilibrium for two main reasons: its convenient analytical structure and because it offers a solution for an equivalent minimization problem. These aspects enable the implementation of the logit model in efficient algorithms to solve the stochastic user equilibrium problem.

Logit-type models were developed for route choice in an attempt to overcome the theoretical drawbacks of the MNL model while maintaining a relatively simple analytical structure. Three models were presented: the C-logit, developed by Cascetta et al. (*1*); the cross-nested logit, developed by Vovsha (*2*); and the PCL, developed by Chu (*3*). The first model purposely was defined for route choice, and the other two were defined as general discrete-choice models. This study adapted both PCL and cross-nested models to the route-choice situation by defining a similarity index and an inclusion coefficient, respectively.

The performance of the models with respect to the overlapping problem was tested for simple networks. As expected, all three models performed better than the MNL model. PCL and cross-nested logit were found to perform better than C-logit, especially in the short bypass–long bypass problem (Example 3).

The cross-nested logit model was found to be very promising, because the loading process can be performed at the link level, but both PCL and C-logit need a route comparison. A preliminary stochastic loading algorithm was presented. For the stochastic user equilibrium problem, the loading phase is performed more efficiently if the inclusion coefficient also is a function of travel costs, because no extra computation is needed to find the shortest *m*-path. The development of a stochastic user equilibrium algorithm with cross-nested logit loading will be addressed in a future report.

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