Stochastic User-Equilibrium Formulations for Extended-Logit Assignment Models

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New stochastic user-equilibrium formulations are presented. In the transportation literature, the "logit assignment" stands for a stochastic user-equilibrium model in which the multinomial logit is the route-choice model. Efficient algorithms using this mathematical formulation were proposed to solve the logit assignment. However, the use of the logit function for route choice has some theoretical drawbacks. In typical transportation networks, many routes have common links, and the structure of the model is not able to account for these common links because the probability for choosing a route is computed based solely on the total route cost. Recently, extended logit-based models were proposed to overcome the overlapping problem and keep the analytical tractability of the logit function. It is demonstrated how extended logit models—such as the Cross-Nested Logit and the Paired Combinatorial Logit—can be derived from more general entropy-type formulations, thus allowing the use of existing (and yet under development) algorithmic solutions for the more general logit-family stochastic assignment model.

The purpose of this paper is to present new stochastic user-equilibrium formulations. *Stochastic user equilibrium* is defined as a situation in which no driver can improve his/her perceived travel costs by unilaterally changing routes. In contrast to the *deterministic approach*, in which all drivers choose the least costly route, the *stochastic approach* implies a selection process among the available routes. In the selection process, routes other than the least costly have a non-zero probability of being chosen.

The stochastic equilibrium models presented in this paper are closely related to recently developed route-choice models. The route-choice models are more general than the simple multinomial logit model in the sense that they can take into account the similarity among routes. The route-choice models presented are logit-based. Therefore, the equilibrium formulations presented can be seen as extensions of the logit assignment.

In the transportation literature, the "logit assignment" stands for a stochastic user-equilibrium model in which the multinomial logit is the route-choice model. Fisk (*1*) developed an equivalent mathematical formulation for the stochastic user-equilibrium problem. Efficient algorithms using this mathematical formulation were proposed to solve the logit assignment (*2*–*5*).

The use of the logit function for route choice has some theoretical drawbacks. The most discussed one is related to the independence of irrelevant alternatives (IIA) property of the logit function. In typical transportation networks, many routes have common links, and the structure of the simple multinomial logit model is not able to account for these common links. The probability of choosing a route depends solely on the total cost of each route in the choice set. Despite this theoretical drawback, many researchers have been motivated to use the logit model for route-choice formulation because of its analytical simplicity; they contend that the congestion effect "alleviates" the overlapping problem in some way.

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Recently, Cascetta et al. (*6*) proposed a modified logit model, named C-Logit, which takes into account the overlapping sections of the routes and keeps the analytical tractability of the logit-family models. Prashker and Bekhor (*7*) presented two other general discrete-choice models of the logit family that also can be adapted for route-choice situations. The models are the Cross-Nested Logit (CNL) model of Vovsha (*8*) and the Paired Combinatorial Logit (PCL) model of Chu (*9*), which was further developed by Koppelman and Wen (*10*). The performance of the extended logit models was tested for simple networks by Prashker and Bekhor (*7*). The adaptation of the Cross-Nested Logit for stochastic equilibrium was suggested by Vovsha and Bekhor (*11*) and is developed further here.

The purpose of this paper is to show how the extended-logit models—such as the Cross-Nested Logit and the Paired Combinatorial Logit—can be derived from more general entropy-type formulations, thus allowing the use of existing (and yet under development) algorithmic solutions for the more general logit-family stochastic assignment model. By presenting the more general formulations and correspondent solutions, this paper introduces a possible wider class of stochastic user-equilibrium formulations for generalized logit models.

This paper is organized as follows: first, Fisk's (*1*) mathematical formulation and the solution are presented for completeness. Next, two extended-logit models are considered: the Cross-Nested Logit model and the Paired Combinatorial Logit model. The paper presents modified entropy-type mathematical formulations and shows that the solutions obtained from these mathematical programs are extended-logit functions. The last part of the paper discusses how the generalized formulations can be implemented in existing algorithms that solve the logit assignment problem.

THE MULTINOMIAL LOGIT MODEL

The solution of Fisk's equivalent stochastic user equilibrium (SUE) minimization program is the logit route-choice model. The mathematical formulation of Fisk's SUE model is stated as follows:

$$\min Z = Z_1 + Z_2$$

$$Z_1 = \sum_a \int_0^{x_a} c_a(w)dw$$

$$Z_2 = \frac{1}{\theta}\sum_{rs}\sum_k f_k^{rs} \ln f_k^{rs}$$

$$\text{s.t.} \sum_k f_k^{rs} = q^{rs} \qquad \forall\ r,s$$

$$f_k^{rs} \geq 0 \qquad \forall\ k,r,s \tag{1}$$

where

f_k^{rs} = flow on route k between origin r and destination s,
c_a = cost on link a,
x_a = flow on link a,
q^{rs} = demand between r and s,
θ = dispersion parameter, and
$f_k \ln f_k = 0$ for $f_k = 0$.

The definitional constraints $x_a = \sum_{rs}\sum_k f_k^{rs}\delta_{ak}^{rs}$ are also part of the problem, where δ_{ak} is equal to 1 for link a on route k and 0 otherwise.

The parameter θ reflects an aggregate measure of drivers' perception of travel costs. The dispersion parameter can be associated with the inverse of the population variance as follows: when the parameter has a high value, the variance is small, indicating that drivers have an acute perception of routes and choose the best ones. The stochastic user-equilibrium solution in this case becomes similar to the deterministic user equilibrium. On the other hand, when the parameter value is low, the variance among drivers is high, indicating that many routes may be chosen, regardless of cost.

To show that this formulation produces the SUE solution, the first-order conditions are developed by forming the Lagrangian function, as follows:

$$L = Z + \lambda^{rs}\left(q^{rs} - \sum_k f_k^{rs}\right) \tag{2}$$

The indexes r and s (origin and destination) have been omitted for simplicity of notation. The partial derivative of L with respect to a path flow f is obtained as follows:

$$\frac{\partial Z_1}{\partial f_k} = \frac{\partial Z_1}{\partial x_a}\frac{\partial x_a}{\partial f_k} = \sum_a c_a\delta_{ak} = c_k \tag{3}$$

$$\frac{\partial Z_2}{\partial f_k} = \frac{1}{\theta}\ln f_k + \frac{1}{\theta} \tag{4}$$

$$\frac{\partial L}{\partial f_k} = \frac{\partial Z}{\partial f_k} - \lambda = c_k + \frac{1}{\theta}\ln f_k + \frac{1}{\theta} - \lambda \tag{5}$$

The solution is obtained by equating the first derivatives to zero:

$$f_k = \exp(\theta\lambda + 1) * \exp(-\theta c_k) \tag{6}$$

Summing the above expression for all routes (k) results in the following expression:

$$\sum_k f_k = \exp(\theta\lambda + 1) * \sum_k \exp(-\theta c_k) = q \tag{7}$$

Combining the two expressions above leads to the probability of choosing a route, which is the simple multinomial logit function:

$$P_k = \frac{f_k}{q} = \frac{\exp(-\theta c_k)}{\sum_k \exp(-\theta c_k)} \tag{8}$$

In the above formulation, all the routes connecting an origin-destination pair should be considered. Akamatsu (*12*) showed that a link-based assignment can be derived if the Markov property holds for the path set. He also showed that Dial's (*13*) STOCH network-loading procedure produces a path set that is consistent with the Markov property.

THE CROSS-NESTED LOGIT MODEL

Model Description

The Cross-Nested Logit model, presented by Vovsha (*8*), was applied for a mode-choice situation. The model was defined as a particular case of McFadden's (*14*, pp.198–272) generalized extreme value (GEV) function. The probability function can be obtained when a generator function $G(y_1, y_2, \ldots, y_n)$ satisfies conditions for serving as a basis of the distribution of random utilities as follows:

1. $G(\ldots)$ is nonnegative.
2. $G(\ldots)$ is homogeneous of degree μ.
3. $\lim_{y_k \to \infty} G(\ldots) = \infty$, for each k.
4. The lth partial derivative of $G(\ldots)$ with respect to any combination of l-distinct y_k's is nonnegative if l is odd and nonpositive if l is even.

When these conditions are satisfied, the probability function for choosing an alternative is given by the following expression:

$$P(k) = \frac{\exp(V_k)\dfrac{\partial G(y_1, y_2, \ldots, y_n)}{\partial y_k}}{\mu G(y_1, y_2, \ldots, y_n)} \tag{9}$$

where $y_k = \exp(V_k)$, with V_k representing the observable components of the utility for each alternative $k(U_k = V_k + \epsilon_k, k = 1, 2, \ldots, n)$.

The generator function for the Cross-Nested Logit model is defined as follows:

$$G(y_1, y_2, \ldots, y_n) = \sum_m \left(\sum_k \alpha_{mk} y_k\right)^{\mu} \tag{10}$$

where

m = nests;
μ = degree of nesting, $0 \le \mu \le 1$; and
α_{mk} = inclusion coefficients allocating alternatives to nests, $0 \le \alpha_{mk} \le 1$.

The inclusion coefficients are subject to a regularity constraint:

$$\sum_m \alpha_{mk} = 1 \tag{11}$$

The probability of choosing an alternative (route) k is then obtained as follows:

$$P(k) = \frac{\exp\left[V_k + \ln\sum_m \alpha_{mk}\left(\sum_l \alpha_{ml}\exp(V_l)\right)^{\mu-1}\right]}{\sum_j \exp\left[V_j + \ln\sum_m \alpha_{mj}\left(\sum_l \alpha_{mj}\exp(V_l)\right)^{\mu-1}\right]} \tag{12}$$

where the utility V_k is assumed to be a linear combination of the path cost c_k.

It is possible to rewrite the expression for choice probability as follows:

$$P(k) = \sum_m P(m)P(k|m) \tag{13}$$

where the conditional probability of a route k being chosen in nest m is

$$P(k|m) = \frac{[\alpha_{mk} \exp(-c_k)]^{1/\mu}}{\sum_l [\alpha_{mk} \exp(-c_l)]^{1/\mu}} \tag{14}$$

and the marginal probability of a nest m being chosen is

$$P(m) = \frac{\left(\sum_k [\alpha_{mk} \exp(-c_k)]^{1/\mu}\right)^{\mu}}{\sum_b \left(\sum_k [\alpha_{bk} \exp(-c_k)]^{1/\mu}\right)^{\mu}} \tag{15}$$

The probability of choosing route k depends on two factors: the generalized cost of the route c_k, and the inclusion coefficient α_{mk} associated with links m that form the route k. The coefficient μ indicates the degree of nesting, as in the Nested Logit model. When $\mu = 1$, the model is equal to the Multinomial Logit. As the degree of nesting becomes higher, $\mu \to 0$, the model becomes probabilistic at the higher (link) level and deterministic at the lower (nest) level.

The adaptation of the CNL model for route choice situations was proposed by Prashker and Bekhor (*7*). It is possible to define a functional relationship for the inclusion coefficient with respect to the links in a route. In the spirit of Cascetta et al. (*6*), this coefficient can be specified as follows:

$$\alpha_{mk} = \left(\frac{L_m}{L_k}\right)^{\gamma} \delta_{mk} \tag{16}$$

where

L_m = link length;
L_k = path length;
δ_{mk} = 1 if link m is on route k and 0 otherwise; and
γ = parameter to be calibrated, which reflects the drivers' perception of similarity among routes. In this paper, we assume for simplicity that $\gamma = 1$.

Equation 16 relates the inclusion coefficients α_{mk} exclusively to the network topology. The CNL model is adapted to route choice by relating the similarity among routes to the physical length of the common sections of the routes. In this way, the defined inclusion coefficients are independent of the travel costs.

The derivation of the CNL model from the GEV class requires the inclusion coefficients to be independent of the generalized cost. If it is assumed that that the inclusion coefficient is proportional to the link costs (instead of link lengths), then α_{mk} is also dependent on congestion. However, this assumption leads to a more complex analytical construct, not only for the route-choice function, but also for the equilibrium formulations presented later in this paper.

The assumption that the inclusion coefficient is independent of congestion means that the similarity effect is treated separately from the congestion effect. However, it should be noted that the assignment model presented later in this paper takes into account the congestion effect.

The formulation of the CNL model presented earlier permits an alternative (for this study, a route) to belong to more than one nest (for this study, a link). The crossing effect is represented by the inclusion coefficients α_{mk}, $0 \le \alpha_{mk} \le 1$. The Nested Logit model is a special case of the Cross-Nested Logit model, in which the coefficients α_{mk} are either zero or 1. By assigning only binary values for α_{mk}, an alternative can only belong to one nest, as in the Nested Logit model.

Figure 1 illustrates how the Nested Logit (NL) and CNL models take into account the overlapping part of a simple network for comparison. The figure also presents the simple Multinomial Logit (MNL) model, which is not capable of taking into account the common links between different routes.

Figure 1 shows a simple network with four links. There are three routes between x and y. Routes 2 and 3 have a common link B. In the MNL model, the probability of choosing a route is dependent only on the total route cost, so it is not possible to "isolate" link B in the tree structure. In the NL model, each route belongs to only one nest. In this simple example, the tree structure is also simple. However, for bigger networks, with many links shared by many routes, the tree structure becomes very complicated. Because each route is restricted to only one nest, the tree representation must "duplicate" the common links to form different routes. Thus, the NL structure cannot solve the overlapping problem in an efficient way.

The tree representation of the CNL model is different from the NL in two points. First, all links are grouped at the upper level of the tree, indicating that each link may belong to different routes. In this way, the tree structure is kept simple, with only two levels. The second point is related to the inclusion coefficient. Because each route may belong to more than one nest (e.g., Route 2 belongs to Nests B and C), the inclusion coefficient represents the proportion of "splitting" the route into the nests. To keep consistency, the inclusion coefficients of each route must sum up to one (e.g., $\alpha_{B2} + \alpha_{C2} = 1$). Using Equation 16 and assuming $\gamma = 1$, we obtain

$$\alpha_{B2} + \alpha_{C2} = \frac{L_B + L_C}{L_2} = 1$$

The following section presents an equivalent stochastic user-equilibrium formulation for the CNL model.

Cross-Nested Logit Equivalent Mathematical Formulation

In this section we show how the CNL model is obtained by developing a stochastic user-equilibrium formulation similar to Fisk's. As shown in the previous section, the CNL is a hierarchical model that can be decomposed into marginal and conditional probabilities. Similarly, the objective function has to be decomposed into two entropy terms, instead of only one, as in the multinomial logit model. In this way, it is possible to obtain the conditional and marginal probabilities as solutions of the equivalent formulation. The following mathematical program is formulated:

$$\min Z = Z_1 + Z_2 + Z_3$$

$$Z_1 = \sum_a \int_0^{x_a} c_a(w)\,dw$$

$$Z_2 = \frac{\mu}{\theta} \sum_{rs} \sum_m \sum_k f_k^{rs} \ln \frac{f_{mk}^{rs}}{(\alpha_{mk}^{rs})^{1/\mu}}$$

$$Z_3 = \frac{1-\mu}{\theta} \sum_{rs} \sum_m \left(\sum_k f_{mk}^{rs}\right) \ln\left(\sum_k f_{mk}^{rs}\right)$$

$$s.t. \sum_m \sum_k f_{mk}^{rs} = q^{rs} \qquad \forall\, r, s$$

$$f_k^{rs} \ge 0 \qquad \forall\, m, k, r, s \tag{17}$$

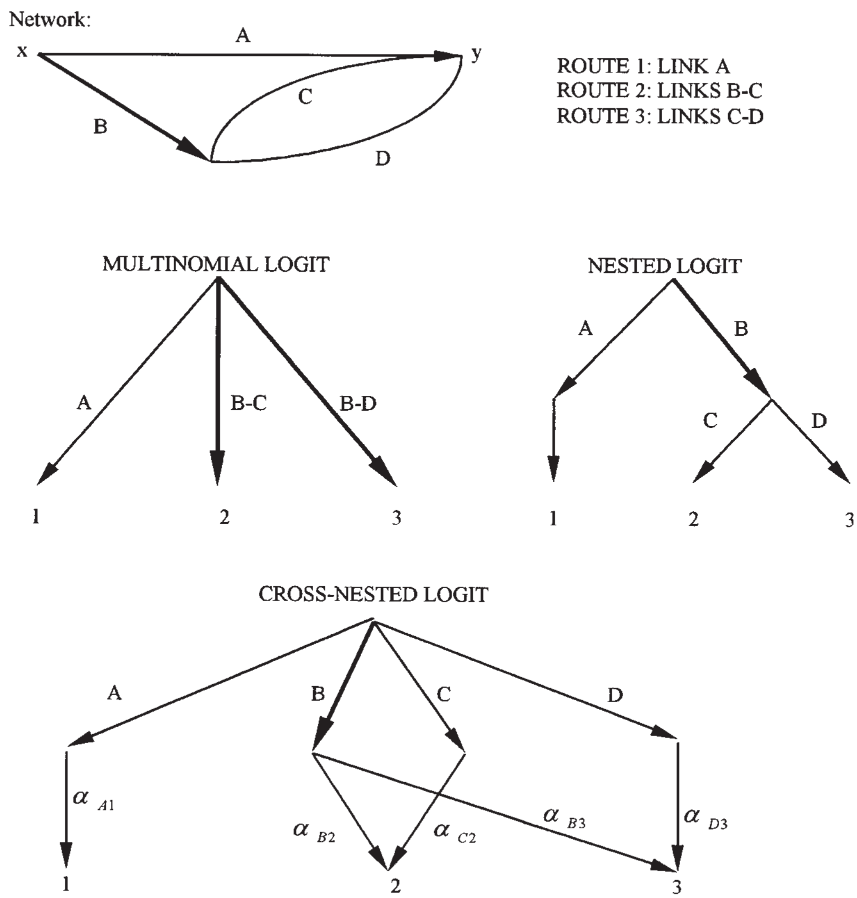

FIGURE 1 Overlapping effect in MNL, NL, and CNL models.

where

f_{mk}^{rs} = flow on path k of nest m between r and s,
q^{rs} = demand between r and s,
c_a = cost on link a,
x_a = flow on link a,
α_{mk}^{rs} = inclusion coefficient of path k in nest m between r and s,
θ = dispersion coefficient,
μ = nesting coefficient, and
$f_{mk}^{rs}\ln\frac{f_{mk}^{rs}}{(\alpha_{mk}^{rs})^{1/\mu}}$ = 0 for either $f_{mk}^{rs} = 0$ or $\alpha_{mk}^{rs} = 0$.

There are two main differences between the mathematical program formulated above and Fisk's: first, the inclusion of another entropy term (Z3), which corresponds to the higher-choice level; and second, the modification of the entropy term (Z2) to include the inclusion coefficient. The summation of the path flows is decomposed by the m links (nests).

The first-order condition is obtained by forming the Lagrangian function for the above formulation. Again, the indexes r and s (origin and destination) are omitted for simplicity of notation. After forming the L function in a similar way as Fisk's, and equating the partial derivatives to zero and multiplying by θ, we obtain the following:

$$\theta c_k + \mu \ln\frac{f_{mk}}{(\alpha_{mk})^{1/\mu}} + (1-\mu)\sum_k f_{mk} + 1 - \theta\lambda = 0 \tag{18}$$

After some manipulations, the following expression is obtained:

$$(f_{mk})\left(\sum_k f_{mk}\right)^{\frac{1-\mu}{\mu}} = \exp[(\theta\lambda - 1/\mu)](\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu] \tag{19}$$

Summing the above expression by route k and elevating both sides by μ result in the following expression:

$$\left(\sum_k f_{mk}\right) = \exp(\theta\lambda - 1)\left\{\sum_k (\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu]\right\}^{\mu} \tag{20}$$

Summing the above expression by nest m gives the following expression:

$$\sum_m\sum_k f_{mk} = q = [\exp(\theta\lambda - 1)]\sum_m\left\{\sum_k (\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu]\right\}^{\mu} \tag{21}$$

Finally, dividing Equation 20 by Equation 21 leads to

$$P(m) = \frac{\sum_k f_{mk}}{q} = \frac{\left\{\sum_k (\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu]\right\}^{\mu}}{\sum_m\left\{\sum_k (\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu]\right\}^{\mu}} \tag{22}$$

which corresponds to the marginal probability of nest m being chosen. To obtain the conditional probability, Equation 18 is divided by Equation 19:

$$f_{mk}\left(\sum_k f_{mk}\right)^{\frac{1-\mu-1}{\mu}} = \frac{(\alpha_{mk})^{1/\mu}\exp(-\theta c_k/\mu)}{\sum_k (\alpha_{mk})^{1/\mu}\exp(-\theta c_k/\mu)} \tag{23}$$

which corresponds to the conditional probability of route k being chosen in nest m:

$$P(k|m) = \frac{f_{mk}}{\sum_k f_{mk}} = \frac{(\alpha_{mk})^{1/\mu}\exp(-\theta c_k/\mu)}{\sum_k (\alpha_{mk})^{1/\mu}\exp(-\theta c_k/\mu)} \tag{24}$$

This concludes the presentation of an equivalent mathematical formulation for the stochastic user-equilibrium problem. The Cross-Nested Logit model was derived as the first-order conditions for the solution of this problem. The next section presents another extended-logit model that can be adapted for route-choice situations, along with the equivalent equilibrium formulation.

PAIRED COMBINATORIAL LOGIT MODEL

Model Description

Another GEV-type model, proposed by Chu (*9*) and later developed by Koppelman and Wen (*10*), was also adapted to model route choice by Prashker and Bekhor (*7*). The generator function $G(\ldots)$ in this case is as follows:

$$G(y_1, y_2, \ldots, y_n) = \sum_{k=1}^{n}\sum_{j=k+1}^{n-1}(1-\sigma_{kj})\left(y_k^{\frac{1}{1-\sigma_{kj}}} + y_j^{\frac{1}{1-\sigma_{kj}}}\right)^{1-\sigma_{kj}} \tag{25}$$

The probability of choosing an alternative (route) k is given as follows:

$$P(k) = \frac{\sum_{j\neq k} e^{\frac{V_k}{1-\sigma_{kj}}}(1-\sigma_{kj})\left(e^{\frac{V_k}{1-\sigma_{kj}}} + e^{\frac{V_j}{1-\sigma_{kj}}}\right)^{-\sigma_{kj}}}{\sum_{l=1}^{n-1}\sum_{m=l+1}^{n}(1-\sigma_{lm})\left(e^{\frac{V_l}{1-\sigma_{lm}}} + e^{\frac{V_m}{1-\sigma_{lm}}}\right)^{1-\sigma_{lm}}} \tag{26}$$

where σ_{kj} is an index of similarity between alternatives k and j.

The double summation includes $n(n-1)/2$ elements, which is the number of different pairs of alternatives in the choice set of n alternatives. If σ_{kj} is equal to zero for all k, j pairs, the PCL collapses to the MNL model. The PCL model allows a differential correlation between pairs of alternatives, as demonsrated by the following. Let

$$P(k) = \sum_{k\neq j} P(kj)\, P(k|kj) \tag{27}$$

where $P(k|kj)$ is the conditional probability of choosing alternative k, given that the binary pair (k,j) was chosen as follows:

$$P(k|kj) = \frac{\exp\left(\frac{V_k}{1-\sigma_{kj}}\right)}{\exp\left(\frac{V_k}{1-\sigma_{kj}}\right) + \exp\left(\frac{V_j}{1-\sigma_{kj}}\right)} \tag{28}$$

and $P(kj)$ is the marginal probability for the binary pair (k, j) as follows:

$$P(kj) = \frac{(1-\sigma_{kj})\left[\exp\left(\frac{V_k}{1-\sigma_{kj}}\right) + \exp\left(\frac{V_j}{1-\sigma_{kj}}\right)\right]^{1-\sigma_{kj}}}{\sum_{l=1}^{n-1}\sum_{m=l+1}^{n}(1-\sigma_{lm})\left[\exp\left(\frac{V_l}{1-\sigma_{lm}}\right) + \exp\left(\frac{V_m}{1-\sigma_{lm}}\right)\right]^{1-\sigma_{lm}}} \tag{29}$$

In the NL model, all pairs of alternatives in a common grouping are required to be similar. In the PCL model, each pair of alternatives can have a similarity relationship that is completely independent of the similarity relationship of other pairs of alternatives. This feature is highly desirable for route-choice models because each pair of routes may have different similarities.

Like the CNL model, which was adapted for route choice by defining the inclusion coefficient, it is possible to relate the similarity index to the network topology. The functional form is similar to that of the C-Logit model as follows:

$$\sigma_{kj} = \left[\frac{L_{kj}}{(L_k L_j)^{0.5}}\right]^{\gamma} \tag{30}$$

where L_{kj} is the length of the common part of routes k and j, γ is a parameter to be calibrated. As with the CNL model, a simple association of the similarity index to the network topology was presented. For simplicity, it is assumed in this paper that $\gamma = 1$.

The above equation confines the similarity-index boundaries between 0 and 1. These conditions have to hold for the PCL model to be consistent with random-utility maximization. If σ_{kj} approaches 1, this indicates that all the links of a path are completely equal to the links of the other path (maximum overlap). On the other hand, if the similarity index is zero, this means that the paths have no link in common (disjointed paths).

Figure 2 illustrates the tree representation of the PCL model for the same simple network example as in Figure 1. In the upper level of the tree representation, the similarity index between the different routes can be calculated with Equation 30. Because there is no link in common between Routes 1 and 2 or between Routes 1 and 3, the similarity index in both cases is zero. Because the model is based on pair comparisons, each route in the lower level is reached by two points from the upper level.

The number of nests in the PCL model increases rapidly with network size because, theoretically, the upper level includes all possible route pairs. However, the tree structure does not change with network size. To implement the PCL model for real-size networks, the number of routes between each O-D pair should be kept small. This is critical in the PCL model because it requires a double summation for each pair of routes in an O-D pair (see Equations 26 or 29) to compute the probability of choosing a path.

The next section shows the development of a stochastic user-equilibrium formulation for which the solution is the Paired Combinatorial Logit model.

Paired Combinatorial Logit Equivalent Mathematical Formulation

The mathematical formulation proposed follows the idea presented for the CNL model. Because the PCL model is also a hierarchical model, the objective function should be composed of two entropy terms: one reflects the higher level (marginal probability of choosing

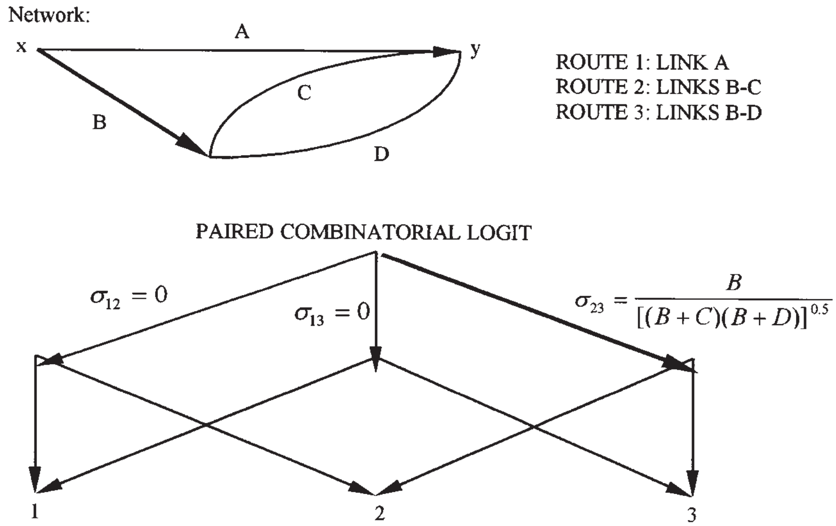

FIGURE 2 Overlapping effect in PCL model.

a pair of routes), and the other reflects the lower level (conditional probability of choosing a route given a chosen pair of routes). The following mathematical formulation is developed:

$$\min Z = Z_1 + Z_2 + Z_3$$

$$Z_1 = \sum_a \int_0^{x_a} c_a(w)\,dw$$

$$Z_2 = \frac{1}{\theta}\sum_{rs}\sum_k\sum_{j\neq k}\beta_{kj}\, f^{rs}_{k(kj)} \ln\left(\frac{f^{rs}_{k(kj)}}{\beta_{kj}}\right)$$

$$Z_3 = \frac{1}{\theta}\sum_{rs}\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}(1-\beta_{kj})\left(f^{rs}_{k(kj)} + f^{rs}_{j(kj)}\right)\ln\left(\frac{f^{rs}_{k(kj)} + f^{rs}_{j(kj)}}{\beta_{kj}}\right)$$

$$s.t. \sum_k\sum_{j\neq k} f^{rs}_{k(kj)} = q^{rs} \qquad \forall\, r,s$$

$$f^{rs}_{k(kj)} \geq 0 \qquad \forall\, k,j,r,s \tag{31}$$

where

$f^{rs}_{k(kj)}$ = flow on route k (of the route pair kj) between r and s;
β_{kj} = measure of the dissimilarity index, defined as $\beta = 1 - \sigma$; and
$f^{rs}_{k(kj)} \ln(f^{rs}_{k(kj)}/\beta_{kj})$ = zero for either $f^{rs}_{k(kj)} = 0$ or $\beta_{kj} = 0$.

Similar to the CNL model, there are two main differences between the mathematical program formulated above and Fisk's: first, the modification of the entropy term (Z2) to include the similarity index in the final expression; and second, the inclusion of another entropy term (Z3), which corresponds to the higher choice level. The summation of the path flows is decomposed by the combinatorial number of pairs.

The solution is obtained in a similar way as the CNL model. To obtain the first-order conditions, the Lagrangian function is formed, and, again, the indexes r and s (origin and destination) are omitted for simplicity of notation. After forming the Lagrangian function, equating the partial derivatives to zero, and multiplying by θ/β, the following equation is obtained:

$$\frac{f_k}{\beta_{kj}}\left(\frac{f_k + f_j}{\beta_{kj}}\right)^{\frac{(1-\beta_{kj})}{\beta_{kj}}} = \exp\left(\frac{\theta\lambda}{\beta_{kj}} - \frac{1}{\beta_{kj}} - \frac{\theta c_k}{\beta_{kj}}\right) \tag{32}$$

For other route j different from k, the following equation holds:

$$\frac{f_j}{\beta_{kj}}\left(\frac{f_k + f_j}{\beta_{kj}}\right)^{\frac{(1-\beta_{kj})}{\beta_{kj}}} = \exp\left(\frac{\theta\lambda}{\beta_{kj}} - \frac{1}{\beta_{kj}} - \frac{\theta c_j}{\beta_{kj}}\right) \tag{33}$$

Dividing Equation 32 by 33 gives

$$\frac{f_k}{f_j} = \frac{\exp\left(\frac{-\theta c_k}{\beta_{kj}}\right)}{\exp\left(\frac{-\theta c_j}{\beta_{kj}}\right)} \tag{34}$$

which gives the binary logit model as the conditional probability of choosing route k from route pair k,j:

$$P(k|kj) = \frac{f_k}{f_k + f_j} = \frac{\exp\left(\frac{-\theta c_k}{\beta_{kj}}\right)}{\exp\left(\frac{-\theta c_k}{\beta_{kj}}\right) + \exp\left(\frac{-\theta c_j}{\beta_{kj}}\right)} \tag{35}$$

where $\beta = 1 - \sigma$ (the similarity index).

To obtain the marginal probability, Equations 32 and 33 are summed and elevated to the power β:

$$(f_k + f_j) = \beta_{kj}\exp(\theta\lambda - 1)\left[\exp\left(-\frac{\theta c_k}{\beta_{kj}}\right) + \exp\left(-\frac{\theta c_j}{\beta_{kj}}\right)\right]^{\beta_{kj}} \tag{36}$$

Summing the above expression for all possible pairs m,l ($m < l$) gives

$$\sum_{m-1}^{n-1}\sum_{l=m+1}^{n}(f_k + f_l) = \exp(\theta\lambda - 1)\sum_{m=1}^{n-1}\sum_{l=m+1}^{n}\beta_{ml}\left[\exp\left(-\frac{\theta c_m}{\beta_{ml}}\right) + \exp\left(-\frac{\theta c_l}{\beta_{ml}}\right)\right]^{\beta_{ml}} \tag{37}$$

Dividing Equation 36 by Equation 37 gives the marginal probability of choosing pair *k,j* among all possible pairs *m,l:*

$$P(kj) = \frac{(f_k + f_j)}{\sum_{m=1}^{n-1}\sum_{l=m+1}^{n}(f_m + f_l)} = \frac{\beta_{kj}\left[\exp\left(-\frac{\theta c_k}{\beta_{kj}}\right) + \exp\left(-\frac{\theta c_j}{\beta_{kj}}\right)\right]^{\beta_{kj}}}{\sum_{m=1}^{n-1}\sum_{l=m+1}^{n}\beta_{ml}\left[\exp\left(-\frac{\theta c_m}{\beta_{ml}}\right) + \exp\left(-\frac{\theta c_l}{\beta_{ml}}\right)\right]^{\beta_{ml}}} \tag{38}$$

This concludes the presentation of how two general logit models can be equivalently formulated in stochastic user-equilibrium formulations. The next section presents some ideas to develop an algorithm for the stochastic-assignment problem using the formulations developed in this paper.

DISCUSSION OF RESULTS

This section discusses some issues regarding the adaptation of stochastic-assignment algorithms to solve the optimization problems presented in this paper. The starting point of the discussion is the well-known Method of Successive Averages (MSA) algorithm.

The MSA algorithm may be used to solve the logit-assignment algorithm, as well as other route-choice models, such as the probit model and the extended-logit models presented in this paper. However, because the equivalent mathematical functions for the logit-type models are obtained in a closed form and the objective function is convex, more efficient algorithms than the MSA can be proposed.

The adaptation of the extended-logit models for route-choice situations demands a path-based stochastic network loading. This is needed because the evaluations of the probability functions for both the CNL and PCL involve path comparisons, which are required for the calculation of the similarity indexes.

Regarding path-based algorithms, there are two basic approaches with respect to the path-set generation. The first approach is to progressively generate paths inside the assignment process, adding new paths to the path set as the iterations of the assignment proceed. This approach is known as *column generation algorithms.* Another approach is to externally define a set of alternative paths based on some criteria. For example, Ben-Akiva et al. (*15*) proposed a “labeling” technique in order to generate paths. This technique was also applied in the work of Cascetta et al. (*16*). The recent work of Cascetta et al. (*17*) can be used to generate a consistent path set based on random-utility choice models.

Damberg et al. (*5*) showed that it is possible to implement efficient path-flow algorithms to solve the assignment problem for large networks. Their algorithm is composed of two main phases: the route-generation phase and the stochastic-assignment phase.

The routes were generated prior to the assignment, based on two main methods. The first method proposed was to find variations of the shortest path between each origin-destination pair. The second method proposed was to perform few deterministic assignment iterations and then store the shortest paths generated at each iteration.

Once the routes are generated and stored, they can be used to compute the probability of using each route in the assignment phase. A stochastic network loading is performed according with the simple logit model, producing a descent direction. A line search is then made in the direction obtained.

Damberg et al. (*5*) pointed out a solution for the overlapping problem between different routes. They proposed different measures of overlap in the route-generation phase. However, this method can be applied only during the route-selection process. In the assignment phase of the algorithm, the simple logit model does not capture the similarity among routes.

In contrast to the simple logit model, the models presented in this paper can capture the similarity among routes during the whole assignment procedure. Damberg et al.’s algorithm could be adapted to the new formulations presented in this paper, with the advantage that, in the assignment phase, a better model can replace the simple logit model.

An algorithm to solve the stochastic user-equilibrium problem with the extended-logit models can be implemented in a similar way to other path-based assignment algorithms, with an additional computational effort caused by the extended entropy terms. The recent works of Damberg et al. (*5*) and Cascetta et al. (*17*) can be used as starting points for such an algorithm.

The probability of choosing a route in the stochastic network-loading phase of the algorithm will be computed in accordance with the extended-logit models. Numerical experiments and performance results are currently being investigated.

SUMMARY

This paper presented two new equivalent formulations for the stochastic user-equilibrium problem. The new formulations were developed based on the relationship between Fisk’s stochastic equilibrium model and the multinomial-logit route-choice model. The formulations were developed by suitably modifying the original entropy term in Fisk’s formulation and adding other entropy terms. The two new formulations obtained are respectively equivalent to two extended-logit route-choice models: the Cross-Nested Logit (CNL) model and the Paired Combinatorial Logit (PCL) model.

The stochastic user-equilibrium formulations presented in this paper were developed by observing the structure of the route-choice models. Both the CNL and the PCL are hierarchical-choice models, and the probability functions can be decomposed into conditional and marginal probabilities. By including two entropy terms representing the hierarchical levels, it was shown that the solutions for the stochastic user-equilibrium problem correspond to the extended-logit models.

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