Link-Nested Logit Model of Route Choice
Overcoming Route Overlapping Problem

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A new link-nested logit model of route choice is presented. The model is derived as a particular case of the generalized-extreme-value class of discrete choice models. The model has a flexible correlation structure that allows for overcoming the route overlapping problem. The corresponding stochastic user equilibrium is formulated in two equivalent mathematical programming forms: as a particular case of the general Sheffi formulation and as a generalization of the logit-based Fisk formulation. A stochastic network loading procedure is proposed that obviates route enumeration. The proposed model is then compared with alternative assignment models by using numerical examples.

A basic assumption for traffic assignment modeling is related to travelers' route choice in the presence of congestion. A behaviorally plausible assumption is that travelers minimize their perceived cost, which is assumed to be a random variable reflecting travelers' imperfect knowledge of travel attributes. This leads to the stochastic user equilibrium model. Sheffi and Powell (*1*) presented a general formulation of stochastic user equilibrium as a mathematical program in which the solution is obtained at the link flow level.

Fisk presented a mathematical program that includes an entropy term for the stochastic user equilibrium (*2*). It is believed that this is the only formulation in which the solution can be obtained at the route level, although this formulation is particular to the multinomial logit route choice model.

To model a solution to this problem it is necessary to define (explicitly or implicitly) a set of available routes for each origin-destination (O-D) pair and evaluate the probability of each route to be chosen given the actual link cost. This inner subproblem is referred to as stochastic network loading.

Existing stochastic network loading models differ mainly on the distribution of the random travel cost. For example, Burrell (*3*) assumed a uniform distribution; Daganzo and Sheffi (*4*) proposed a normal distribution leading to a probit-based route choice model and a Monte-Carlo simulation technique for probability calculation.

Dial (*5*) proposed a stochastic multipath loading procedure for a multinomial logit model of route choice that obviates route enumeration. Another implicit method suggested in the literature is to simulate link costs. This solution is criticized by other authors (*6*). In the branch of explicit route enumeration, progress has been made in effective choice set generation procedures, although it is based on some heuristics (*7,8*).

The main disadvantage of the multinomial logit model is the well-known IIA property that leads to erroneous loading on overlapping routes. The recently proposed C-logit model tries to overcome the route overlapping problem, keeping the analytically convenient logit structure of the route choice model (*9*). This is achieved at the expense of explicit choice set formation. Also, the structure of the C-logit model cannot be rigorously derived from the assumption on random cost distribution in the framework of models with a constant choice set.

In this report, the recently proposed cross-nested logit model (*10*) is extended to model route choice.

CROSS-NESTED LOGIT MODEL

The cross-nested logit model is a member of the generalized-extreme-value (GEV) class. It constitutes a generalization of the nested logit model, but it allows choice alternatives to belong to more than one nest in flexible proportions, reflecting differential similarities among them. The GEV modeling class was proposed by McFadden (*11*) and is formulated in the following way. Let $r = 1, 2, \ldots R$ denote the discrete choice alternatives and $G(y_1, y_2, \ldots y_r, \ldots y_R)$ be a generator function of R nonnegative variables with the following properties:

1. $G(\ldots)$ is nonnegative and homogeneous of degree 1.
2. $\lim_{y_r \to \infty} G(\ldots) = \infty$, for each r.
3. The lth partial derivative of $G(\ldots)$ with respect to any combination of l distinct y_r, is nonnegative if l is odd and nonpositive if l is even.

By denoting $G_r(\ldots) = \partial G(\ldots)/\partial y_r$, we obtain the GEV model for the choice probability of alternative r as

$$P(r) = \frac{e^{-c_r} G_r\left(e^{-c_1}, e^{-c_2}, \ldots e^{-c_r}, \ldots e^{-c_R}\right)}{G\left(e^{-c_1}, e^{-c_2}, \ldots e^{-c_r}, \ldots e^{-c_R}\right)} \quad (1)$$

where c_r denotes an observed component of generalized cost of alternative r.

The cross-nested logit model is derived from the GEV class if the generator is specified as

$$G = \sum_a \left(\sum_r \alpha_{ar}^{1/\mu} y_r^{1/\mu} \right)^{\mu} \quad (2)$$

where

a = nests,
μ = degree of nesting, $0 < \mu \le 1$; and
α_{ar} = inclusion coefficients allocating alternatives to nests, $0 \le \alpha_{ar} \le 1$.

Inclusion coefficients are subject to a regularity constraint:

$$\sum_a \alpha_{ar} = 1 \quad (3)$$

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The cross-nested logit model was described in the mode choice context (*10*), where it was proved that the generator of Equation 2 meets all the above-mentioned GEV requirements. A restricted case of the cross-nested structure, in which nests were organized by order of alternatives, has been applied by Small (*12*) for modeling arrival-time choice. It was shown that the generator of Equation 2 produced a flexible and analytically convenient expression for choice probability (*10*,*12*):

$$P(r) = \sum_a P(a)P(r|a) \tag{4}$$

where a conditional probability of an alternative *r* being chosen in the nest *a* is

$$P(r|a) = \frac{\alpha_{ar}^{1/\mu} e^{-c_r/\mu}}{\sum_s \alpha_{as}^{1/\mu} e^{-c_s/\mu}} = \frac{e^{(\ln \alpha_{ar} - c_r)/\mu}}{\sum_s e^{(\ln \alpha_{as} - c_s)/\mu}} \tag{5}$$

and a marginal probability of a nest *a* being chosen is

$$P(a) = \frac{e^{-\tilde{c}_a}}{\sum_b e^{-\tilde{c}_b}} = \frac{\left(\sum_r \alpha_{ar}^{1/\mu} e^{-c_r/\mu}\right)^\mu}{\sum_b \left(\sum_r \alpha_{br}^{1/\mu} e^{-c_r/\mu}\right)^\mu} = \frac{\left(\sum_r e^{(\ln \alpha_{ar} - c_r)/\mu}\right)^\mu}{\sum_b \left(\sum_r e^{(\ln \alpha_{br} - c_r)/\mu}\right)^\mu} \tag{6}$$

where $\tilde{c}_a$ denotes the composite cost of the nest *a*. This reflects the costs of the alternatives included in this nest as well as their inclusion coefficients.

The rationale of the cross-nested structure versus the ordinary nested logit model is that alternatives can be allocated to nests ambiguously, reflecting a differential degree of similarity between them (for example, an alternative can be proportionally allocated to two nests: 30 percent/70 percent), whereas the ordinary nested structure allows only for groupwise specification of similarities based on an unambiguous allocation of alternatives to nests.

The cross-nested logit model has the following useful properties (*10*):

1. If $\mu = 1$, the generator of Equation 2 regarding the regularity constraint of Equation 3 takes the form

$$G = \sum_r y_r$$

and the model collapses to a multinomial logit form.

2. If $\alpha_{ar} = (0,1)$, that is, alternatives are unambiguously allocated to nests and each nest *a* has its own subset of alternatives R_a, then the generator of Equation 2 takes the form

$$G = \sum_a \left(\sum_{r \in R_a} y_r^{1/\mu}\right)^\mu$$

and the model collapses to the nested logit form. This case is described in greater detail by Ben-Akiva and Lerman (*13*).

3. Property 3 is common to all GEV constructs. If a scale unit of cost (implicitly incorporated into expression c_r) becomes infinitely small—the absolute differences between costs for different alternatives become infinitely large—the model collapses to the deterministic case in which the cheapest alternative is chosen.

4. If $\mu \to 0$, the model collapses to the multinomial logit form at the higher level (level of nests) and to the deterministic form at the lower level (level of alternatives). Formally the expression for the composite nest utility and expressions for Conditional 5 and Marginal 6 probabilities can be rewritten in the following way:

$$\lim_{\mu \to 0} -\tilde{c}_a = \lim_{\mu \to 0} \ln\left(\sum_r e^{(\ln \alpha_{ar} - c_r)/\mu}\right)^\mu = \max_r(\ln \alpha_{ar} - c_r) = \ln \alpha_{ar(a)} - c_{r(a)} \tag{7}$$

$$\lim_{\mu \to 0} P(r|a) = \begin{cases} \frac{1}{|R(a)|}, & \text{if} \quad r \in R(a), \\ 0, & \text{if} \quad r \notin R(a) \end{cases} \tag{8}$$

$$\lim_{\mu \to 0} P(a) = \frac{e^{\ln \alpha_{ar(a)} - c_{r(a)}}}{\sum_b e^{\ln \alpha_{br(b)} - c_{r(b)}}} \tag{9}$$

where $r(a)$ is the *a*-nest best (chosen) alternative. If there is a tie between several equal *a*-nest best alternatives, we use the notation $R(a)$ for the set.

5. If $\mu \to 0$ and we add a new alternative that is identical to an existing alternative for cost and inclusion coefficients (in other words, one of the alternatives is artificially divided into two equal and totally duplicating alternatives), the probability of this duplicated alternative will be shared equally with the new alternative, whereas the probabilities of other alternatives will not change (in line with the "red bus–blue bus" problem of mode choice, which also is resolved by means of the maximum degree of nesting in the ordinary nested logit model).

6. The expected value of the composite cost by alternatives in a choice set (i.e., the average cost experienced by a user) can be written for the cross-nested logit model as for a member of the GEV family in the following form (*11*):

$$E\left[\max_r(-C_r)\right] = -E\left(\min_r C_r\right) = \ln G(\ldots) = \ln \sum_a \left(\sum_r \alpha_{ar}^{1/\mu} e^{-c_r/\mu}\right)^\mu = \ln \sum_a e^{-\tilde{c}_a} \tag{10}$$

where C_r denotes a random cost associated with a choice of alternative *r* including both an observed component and a random disturbance. An expression for $G(\ldots)$ is defined for the cross-nested logit model by Equation 2.

7. If one of the nests is divided artificially into, for example, two identical nests comprising the same subset of alternatives but with inclusion coefficients equal to half of their previous values, then the final choice probabilities given by Formula 4 will be the same for all alternatives. This property can be generalized in a case in which the new nests are organized in some other proportion or in which the number of new nests is greater than two. This important property of the cross-nested logit model means that logically equivalent transformations of the nesting structure do not affect the choice probabilities. More formally, a nesting structure with nests *m* is equivalent to a more detailed nesting structure with nests *n* obtained by subdivision of the nests *m*, if inclusion coefficients are calculated as $\alpha_{nr} = \alpha_{mr} \times \eta_n$ where

$$\sum_{n \in N_m} \eta_n = 1$$

and N_m denotes a subset of nests *n* obtained from the nest *m*.

8. A cross-elasticity of choice probability of an alternative with respect to the generalized cost of another alternative depends on the inclusion of these two alternatives into the same nests. All else being equal, if alternatives are included into completely different nests, the cross-elasticity will be minimal (random cost components will be independent). The more alternatives included in the same nests, the greater the cross-elasticity will be. In the extreme case, if two alternatives have equal sets of inclusion coefficients, the cross-elasticity will be maximum. This situation is analogous to inclusion of alternatives in the same nest in an ordinary nested structure.

LINK-NESTED LOGIT MODEL OF ROUTE CHOICE

Route choice is a special case of discrete choice in which the degree of similarity between routes can be measured naturally as spatial overlap (*9,14*). This consideration gives rise to a nesting structure by links, in which each relevant link *a* is considered a nest and inclusion coefficients reflect a proportion of route length (or time) spent on each link:

$$\alpha_{ar} = \frac{L_a}{L_r}\delta_{ar} \qquad (11)$$

where

L_a = length (time) of link *a*,
L_r = length (time) of route *r*, and
δ_{ar} = link-route incidence coefficient, $0 < \delta_{ar} < 1$.

In the route choice context the special term *link-nested model* is used instead of the more general term *cross-nested model.* The modeling considerations—derivation of the link-nested model from the GEV class and further inclusion of the route choice model into an overall stochastic user equilibrium context— require that inclusion coefficients be constant (independent of costs which vary with route loading). Relaxation of this requirement to make congested travel times available as a link attribute is possible but leads to a more complex analytical construct, which has yet to be explored.

From this point of view any link attribute like physical length or free-flow time can be used only if it reflects link impedance in a behavioral sense and if it is link-additive for the route. This subjects inclusion coefficients to a regularity Constraint 3. This attribute also can be weighted reflecting road-link hierarchy. The final decision of which link attribute is most suitable for route overlap calculation can be made after calibration by using real-world networks.

An interesting behavioral issue is whether route overlap as a measure of route similarity could be (or should be) calculated independently of congestion considerations. It appears reasonable to calculate route overlap independently of congestion travel times, thus basing this measure exclusively on the network configuration and not on the demand matrix. However, congested links may be given larger weights within a route choice framework.

It should be noted that for uncongested network fragments (to which the whole issue of stochastic loading primarily is aimed), both types of link attributes (congestion-dependent and congestion-independent) perform in similar ways. A free-flow time is employed in this report as the link attribute.

Another issue that allows for some flexibility is the definition of a set of relevant links for each O-D pair. In practice, the usual considerations for route choice set generation will be appropriate for identifying relevant links (*6,7,9*). However, the principal difference here is that in the proposed modeling framework only an explicit set of relevant links is required, whereas a route choice set is formed implicitly by using Property 4 from the previous section.

Next, the formulation of the eight properties in the context of route choice is refined.

Property 1 means that a known logit model and effective loading procedure developed for it constitute a particular case of the link-nested model with the minimal degree of nesting ($\mu = 1$). The problem of route overlapping cannot be resolved by a simple logit model and a particular case of the link-nested model with a maximum degree of nesting ($\mu \to 0$) will be used. The maximum degree of nesting is meaningful for route choice, because completely (or almost completely) overlapping routes should be considered a single (or almost single) option from the behavioral standpoint.

Property 2 is helpful for understanding the general cross-nested model formulation, but it is meaningless in the route choice context because the ordinary nested structure based on unambiguous route grouping cannot be built in a general case of route choice.

Property 3 means that if one route is much cheaper than all others in terms of absolute cost differential, this route will be chosen exclusively regardless of nesting structure (the route choice degenerates to a deterministic case).

Property 4 is of crucial importance for substantiation of the stochastic network loading procedure that obviates route enumeration. By this property, in the case of maximum degree of nesting ($\mu \to 0$), only the cheapest route is chosen for each link-nest and must be loaded according to link-nest composite cost given by Formula 9. This formula can be interpreted in the following way: The probability of choosing a link-nest *a* is the probability of the *a*-cheapest route that passes through the link *a,* to be the cheapest route among all the routes.

Property 5 states that if $\mu \to 0$ and a new route is added that duplicates the existing route (or stated otherwise, the route is artificially divided into two totally overlapping routes), this will not change probabilities of any but the duplicated route. The probability of the latter will be shared equally with the new route. This consideration serves as a logical substantiation of a maximum degree of nesting regardless of the computational advantages of loading without route enumeration. Of course, this is only a particular desirable property of the route choice model. A final conclusion on the validity of the maximum degree of nesting can be made only following calibration of the model on real-world observations.

Property 6 gives a simple closed analytical expression for average cost expected by the user (route chooser). It will be employed in the formulation of the stochastic user equilibrium program incorporating the link-nest route choice model.

Property 7 is of great importance and is desirable for any route choice model because it shows model insensitivity to formal network description. If links are divided or united artificially, keeping the network in the same shape from the traffic assignment point of view, the link-nested model yields the same route choice probabilities.

Property 8 also is a basic desirable feature of any route choice model. It means that for a pair of routes, the more common links (overlaps) they have and the longer these common links are, the more these routes are similar (have correlated random cost components) and hurt each other with respect to choice probabilities. However, nonoverlapping routes that have no common link are treated as in the ordinary multinomial logit model. In particular, when a choice set includes disjoint routes only, the link-nested model will perform exactly like the multinomial logit model regardless of the degree of nesting.

Simplified Numerical Examples

Following are two examples in which a single trip is loaded onto an uncongested network. Link lengths (measured in kilometers) are equal to link times (measured in minutes). Cost should equal total route time in minutes multiplied by a scaling parameter equal to 0.1. Routes are numbered according to the serial numbers of included links.

Example 1

The first example is the blue route–red route problem. This issue refers to a known drawback of the multinomial logit model of route choice. This results in overloading of the network in the more dense fragments as a result of route multiplication although the routes are heavily overlapped (*6,14,15*).

Consider a single O-D pair connected by a four-link network, as shown in Figure 1. There are three possible routes, two of them having a common link 2. Consequently, this link is a single link-nest that has two routes, whereas all other link-nests have one route each. A pair of overlapping routes, 23 and 24, can be referred to as blue route–red route to stress their similarity.

Retaining the same basic spatial configuration and the total length for each of the three routes, a degree of overlap is changed successively between Routes 23 and 24—the proportion between link 2 and links 3 and 4, as shown in Figure 1. Case 1 can be characterized as a heavy overlap of 90 percent where Routes 23 and 24 represent practically the same travel option but either are blue or red. Case 2 is a balanced situation in which the degree of overlap is 50 percent. Finally, Case 3 depicts an opposite extreme situation in which the degree of overlap is comparatively small (10 percent) and all three routes represent distinct travel options.

The multinomial logit model cannot distinguish between these cases and yields the same trivial result of equal probabilities for all three routes (recalling that the cost for all of them is the same and equal to $10 * 0.1 = 1$). At the network level it leads to a relative overloading of link 2, which is common to two of the routes.

The link-nested logit model can treat differential degrees of overlap. In Case 1, Route 1 obtains almost the same probability (0.48) as the two overlapping Routes 23 and 24 taken together (0.52), the latter two receiving equal probabilities of 0.26. In Case 2, the link-nested model assigns 0.4 trips to Route 1 and 0.6 trips to Routes 23 and 24 together, reflecting the comparatively wider scope of behavioral opportunities on this pair of routes. In Case 3, the link-nested model approaches the multinomial logit model, reflecting the fact that all three routes have become distinct travel options.

Formally, the desired flexibility of choice probabilities is achieved in each case by using the same link-nested structure by means of differential inclusion of routes into link-nests. In Case 1 the nest of link 2 contains 90 percent of Routes 23 and 24, so that these routes are extremely competitive. In contrast, in Case 3 the nest of link 2 contains only 10 percent of Routes 23 and 24, and all other link-nests contain one route each. Thus, the model approaches the simple multinomial logit structure.

This also can be resolved by an appropriately calibrated C-logit model (*9*) with similar numerical results.

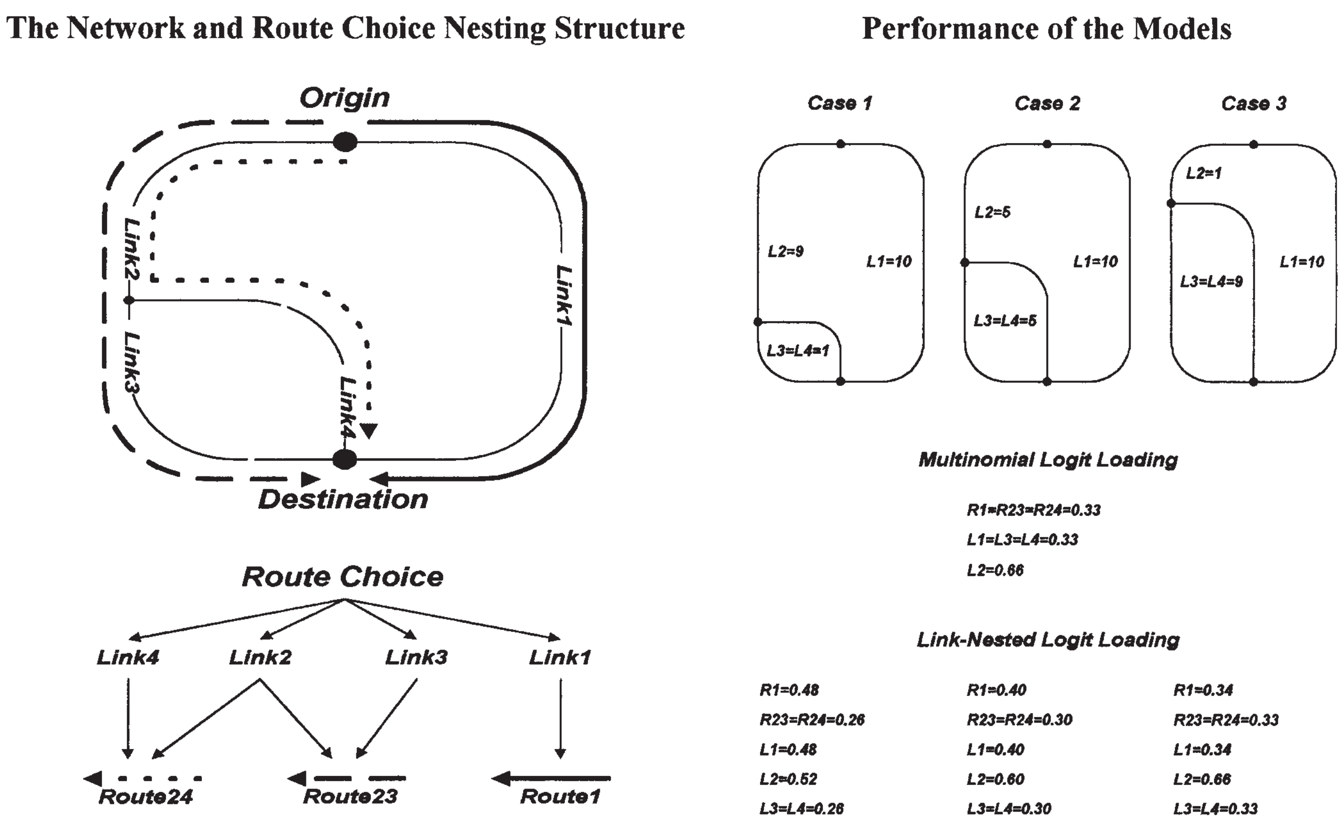

FIGURE 1 **Blue route–red route problem.**

Example 2

The second example is known as the short bypass–long bypass problem. This issue refers to another well-known problem associated with the multinomial logit: its inability to distinguish between different routes with a relatively small overlap (considered by drivers to be distinct options) and artificially divided routes that almost duplicate one another.

Consider again a single O-D pair connected by a four-link network but with a different configuration, as shown in Figure 2. There are two possible routes, having common links 1 and 4. Consequently, these link-nests have two routes each, whereas nests of links 2 and 3 have one route each.

As in Example 1, the same basic spatial configuration and total length for each of the two routes is retained, but the degree of overlap between them is successively changed, by changing the proportion between links 1 and 4, on the one hand, and links 2 and 3, on the other hand. Case 1 can be characterized as exhibiting heavy overlap—90 percent of Route 124 (9 out of 10 km) and 82 percent of Route 134 (9 out of 11 km). In this case the routes represent practically the same travel option. Case 2 is a balanced situation in which the degree of overlap is 60 percent of Route 124 and 55 percent of Route 134. Finally, Case 3 depicts the contrasting situation in which the degree of overlap is comparatively small (30 percent of Route 124 and 27 percent of Route 134), so the routes represent distinct travel options.

The multinomial logit model again cannot distinguish between these cases and yields constant probabilities across cases stemming from the difference in costs: 0.52 for Route 124 with cost equal to $10 * 0.1 = 1$, versus 0.48 for Route 134 with cost equal to $11 * 0.1 = 1.1$.

Once again, the link-nested logit model is sensitive to the differential degree of overlap. In Case 1, Route 124 obtains the dominant probability (0.86) against Route 134 (0.14), properly reflecting the behavioral assumption that drivers will consider the bypass Route 134 as a redundant option with longer travel time. In Case 2 the link-nested model assigns 0.71 trips to Route 124 and 0.29 trips to Route 134, reflecting the greater scope of behavioral opportunities on this pair of routes. In Case 3 the link-nested model approaches the multinomial logit model reflecting the fact that the two routes have become distinct travel options.

As before, the desired flexibility of choice probabilities is achieved by using an identical link-nested structure in each case, by means of differential inclusion of routes into link-nests. In Case 1 the nests of links 1 and 4 taken together contain 90 percent of Route 124 and 82 percent of Route 134, thus making these routes extremely competitive. By contrast, in Case 3 these nests contain only 30 percent of Route 124 and 27 percent of Route 134, and other link-nests contain one route each, so that the model approaches the simple multinomial logit structure.

It is interesting to note that this network structure does not lend itself to resolution by the C-logit model. Indeed, the C-logit model computes choice probabilities according to the formula (*9*):

$$P(r) = \frac{e^{-c_r - cf_r}}{\sum_{s \in R} e^{-c_s - cf_s}} \tag{12}$$

where cf_r is a commonality factor that reduces the attractiveness of a route r because of overlapping routes. The commonality factor is calculated by the following formula:

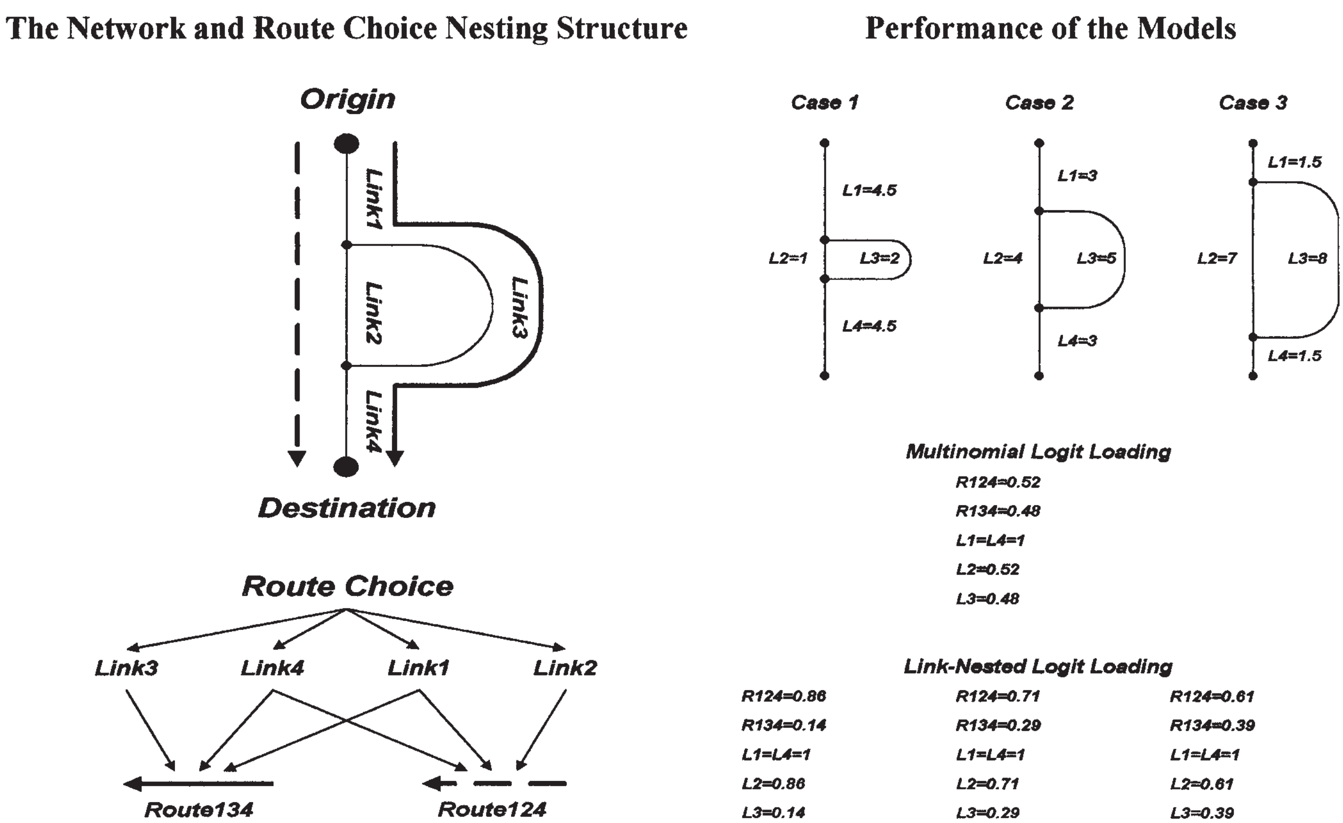

FIGURE 2 Short bypass–long bypass problem.

$$cf_r = \beta \ln \sum_{s \in R} \left(\frac{L_{rs}}{L_r^{1/2} L_s^{1/2}} \right)^{\gamma} \tag{13}$$

where L_{rs} is the length of the overlap of routes r and s and β, γ are parameters for calibration.

Formula 13 can be rewritten equivalently in terms of inclusion coefficients in Equation 11:

$$cf_r = \beta \ln \sum_{s \in R} \left(\sum_a \alpha_{ar}^{1/2} \alpha_{as}^{1/2} \right)^{\gamma} \tag{14}$$

In Example 2, the commonality factor will be equal in each case for both routes (although its absolute value changes between cases). Consequently, the C-logit model predicts the same constant choice probabilities as the ordinary multinomial logit model.

STOCHASTIC USER EQUILIBRIUM: PROGRAM FORMULATION

The following notations are used in the program formulation of stochastic user equilibrium:

p = origins,
q = destinations,
d^{pq} = demand between O-D pair pq,
r = routes,
a = links as nests in the link-nested structure,
b = links as network objects for loading,
x_{ar}^{pq} = route loading detailed by the higher-level nests (links),
$x_r^{pq} = \sum_a x_{ar}^{pq}$ = route loading,
$y_b = \sum_{pqr} \delta_{br} x_r^{pq}$ = link flows, and
c_b = observed cost on link b as a function of link flow y_b, assumed to be positive, strictly increasing, and twice continuously differentiable.

A mathematical program is developed of stochastic user equilibrium for the link-nested logit model of route choice in two different ways: as a restricted case of a general formulation given by Sheffi (*14*) and as a generalization of the logit-based formulation given by Fisk (*2*).

The Sheffi formulation is devised by substituting the general expression for average cost expected by a user with the special expression for the link-nested logit model in Equation 10. This yields the following unconstrained program:

$$\min\ Z(\mathbf{x}, \mathbf{y}) = Z_1 + Z_2 + Z_3 \tag{15}$$

where

$$Z_1 = -\sum_{pq} d^{pq} E\left(\min_r C_r^{pq} \right) = \sum_{pq} d^{pq} \ln \sum_a \left(\sum_r \alpha_{ar}^{1/\mu} e^{-c_r^{pq}/\mu} \right)^{\mu} \tag{16}$$

$$Z_2 = \sum_b c_b y_b \tag{17}$$

$$Z_3 = -\sum_b \int_0^{y_b} c_b(w)\,dw \tag{18}$$

A sufficient assumption for Sheffi's program is that the distribution of the average expected cost be translationary invariant (*14,16*), which holds for choice models of the GEV class (*13*). It has been shown by Williams (*17*) that the derivative of the expected cost with respect to a systematic component of the cost of a particular route is equal to choice probability of this route:

$$\frac{\partial E\left(\min_r C_r^{pq} \right)}{\partial c_r^{pq}} = P^{pq}(r) = \sum_a P^{pq}(a) P^{pq}(r|a) \tag{19}$$

In the particular case of the link-nested logit model of route choice, this formula can also be independently derived by strict differentiation of Expression 10:

$$\begin{aligned} \frac{\partial \ln \sum_a \left(\sum_r \alpha_{ar}^{1/\mu} e^{-c_r^{pq}/\mu} \right)^{\mu}}{\partial c_r^{pq}} &= \frac{1}{\sum_a (\ldots)^{\mu}} \left[\sum_a \mu (\ldots)^{\mu-1} \right] \alpha_{ar}^{1/\mu} e^{-c_r^{pq}/\mu} \frac{1}{\mu} \\ &= \sum_a \frac{(\ldots)^{\mu-1}}{\sum_a (\ldots)^{\mu}} \frac{(\ldots)}{(\ldots)} e^{-c_r^{pq}/\mu} = \sum_a \frac{(\ldots)^{\mu}}{\sum_a (\ldots)^{\mu}} \frac{e^{-c_r^{pq}/\mu}}{(\ldots)} \\ &= \sum_a P^{pq}(a) P^{pq}(r|a) \end{aligned} \tag{20}$$

It also was shown that although the objective function of the program in Equations 15 through 18 is nonconvex globally, there is only one stationary point at which the objective function is strictly convex locally in the link flows, which hence is the unique stochastic equilibrium set of link flows (*14,16*). The first-order condition can be written by using the following expressions for the first derivatives of the objective function components (making allowance for Equation 20):

$$\frac{\partial Z_1}{\partial y_b} = \sum_{pqr} d^{pq} \sum_a P^{pq}(a) P^{pq}(r|a) \delta_{br} \frac{\partial c_b}{\partial y_b} \tag{21}$$

$$\frac{\partial Z_2}{\partial y_b} = c_b + \frac{\partial c_b}{\partial y_b} y_b \tag{22}$$

$$\frac{\partial Z_3}{\partial y_b} = -c_b \tag{23}$$

After combining the expressions of Equations 21 through 23 and straightforward simplifications, the first-order condition can be expressed as

$$\sum_{pqr} d^{pq} \sum_a P^{pq}(a) P^{pq}(r|a) \delta_{br} = y_b \tag{24}$$

This set of link flows exactly matches route loading by the stochastic user equilibrium condition: Expression 24 can be reproduced if we write link flow y_b as a function of route loadings,

$$y_b = \sum_{pqr} x_r^{pq}\, \delta_{br}$$

where route loadings are calculated by the route choice model

$$x_r^{pq} = d^{pq} P^{pq}(r)$$

It should be noted, however, that although the solution of this program produces a unique set of link flows, there may be more than one solution for equilibrium route loadings.

Next another mathematical program, which also corresponds to stochastic user equilibrium with the link-nested route choice model, is developed but this time as a generalization of Fisk's logit-based formulation (*2*). This program formulation has an advantage of strict convexity in both the link flow and route loading variables, and consequently a unique set of route loadings is provided. The following constrained mathematical program is formulated:

$$\min Z(\mathbf{x}, \mathbf{y}) = Z_1 + Z_2 + Z_3$$

$$\text{s.t.} \sum_{ar} x_{ar}^{pq} = d^{pq}, \quad (u^{pq}) \tag{25}$$

where

$$Z_1 = \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} \sum_b \int_0^{y_b} c_b(w)dw \tag{26}$$

is a route cost-formation term,

$$Z_2 = \frac{1}{\theta_1} \sum_{pqar} x_{ar}^{pq} \ln \frac{x_{ar}^{pq}}{\alpha_{ar}^{pq}} \tag{27}$$

is an entropy term corresponding to a lower (route) choice level,

$$Z_3 = \frac{1}{\theta_2} \sum_{pqa} \left(\sum_r x_{ar}^{pq} \right) \ln \left(\sum_r x_{ar}^{pq} \right) \tag{28}$$

is an entropy term corresponding to a higher (link-nest) choice level, and $\theta_1 > 0$, $\theta_2 > 0$ are parameters.

The main difference of the program in Equations 25 through 28 from Fisk's original logit-based formulation (*2*) is that in the original formulation only one entropy term appears, reflecting the simple multinomial structure of the route choice model. The link-nested logit model requires that two entropy terms be introduced, reflecting the hierarchical nesting structure of route choice.

The expressions for first-order derivatives of the objective function components can be readily derived:

$$\frac{\partial Z_1}{\partial x_{ar}^{pq}} = \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} \sum_b c_b \delta_{br} = \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} c_r^{pq} \tag{29}$$

$$\frac{\partial Z_2}{\partial x_{al}^{pq}} = \frac{1}{\theta_1} \ln \frac{x_{ar}^{pq}}{\alpha_{ar}^{pq}} + \frac{1}{\theta_1} \tag{30}$$

$$\frac{\partial Z_3}{\partial x_{al}^{pq}} = \frac{1}{\theta_2} \ln \sum_r x_{ar}^{pq} + \frac{1}{\theta_2} \tag{31}$$

Forming the Lagrangian function

$$L(\mathbf{x}, \mathbf{y}, \mathbf{u}) = Z(\mathbf{x}, \mathbf{y}) + \sum_{pq} u^{pq} \left(d^{pq} - \sum_{ar} x_{ar}^{pq} \right)$$

and equating to zero derivatives of the Lagrangian function by route loading variables **x**, we obtain the following:

$$\frac{1}{\theta_1} \ln \frac{x_{ar}^{pq}}{\alpha_{ar}^{pq}} + \frac{1}{\theta_2} \ln \sum_r x_{ar}^{pq} + \frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{\theta_1 + \theta_2}{\theta_1 \theta_2} c_r^{pq} - u^{pq} = 0 \tag{32}$$

and after equivalent transformation

$$x_{ar}^{pq} \left(\sum_r x_{ar}^{pq} \right)^{\theta_1 \theta_2} = \alpha_{ar}^{pq} e^{-1-\theta_1/\theta_2 + u^{pq} - \frac{\theta_1+\theta_2}{\theta_2} c_r^{pq}} = \tilde{d}^{pq} \alpha_{ar}^{pq} e^{-\frac{\theta_1+\theta_2}{\theta_2}} \tag{33}$$

Summing Equation 33 by *r*

$$\left(\sum_r x_{ar}^{pq} \right)^{\frac{\theta_1+\theta_2}{\theta_2}} = \tilde{d}^{pq} \sum_r \alpha_{ar}^{pq} e^{-\frac{\theta_1+\theta_2}{\theta_2} c_r^{pq}} \tag{34}$$

and denoting

$$\mu = \frac{\theta_2}{\theta_1 + \theta_2}$$

we obtain an expression for aggregate route loading:

$$\sum_r x_{ar}^{pq} = (\tilde{d}^{pq})^{\mu} \left(\sum_r \alpha_{ar}^{pq} e^{-c_r^{pq}/\mu} \right)^{\mu} = \tilde{D}^{pq} e^{-\tilde{c}_a^{pq}} \tag{35}$$

Then combining this with Constraint 25 we obtain an expression for $\tilde{D}^{pq}$:

$$\tilde{D}^{pq} = \frac{d^{pq}}{\sum_a e^{-\tilde{c}_a^{pq}}} \tag{36}$$

Substituting Expression 36 into Equation 35 we obtain the expression for $P^{pq}(a)$, which exactly matches the formula for marginal probability of the cross-nested logit model of Equation 6. Substituting this expression into the basic formula (Equation 33), it can be shown that an expression for $P^{pq}(r|a)$ also exactly suits the formula for conditional probability of the cross-nested logit model (Equation 5).

The mathematical program of Equations 25 through 28 can be solved by an adaptation of the linear approximation algorithm of Frank and Wolfe, which is a suitable tool for equilibrium assignment objective functions with entropy demand terms. This issue deserves special attention and is beyond the scope of this report. Thus, the method of successive averages is employed (*1*), which suffices for investigation of the model properties.

STOCHASTIC NETWORK LOADING WITHOUT ROUTE ENUMERATION

The following procedure has been developed for an extreme case of the maximum degree of nesting when $\mu \to 0$. A generalization is possible for the common case of intermediate degree of nesting where $0 < \mu < 1$ (if $\mu = 1$, the model collapses to the multinomial logit form). However, this generalization requires route enumeration and explicit calculation of logsum for the composite cost of each link-nest. The maximum degree of nesting in route choice also can be justified from the behavioral standpoint, at least in the case of heavily overlapped routes.

The general framework of the assignment algorithm can be summarized as follows:

1. Initialization: Construct all-node shortest path with free-flow times. Perform subroutine Nestload, yielding initial link flows $\mathbf{y}^{\theta}$.
2. Update: Update iteration counter $n = n + 1$. Update link costs, $\mathbf{c} = c(\mathbf{y})$.
3. Determine direction: Construct all-node shortest path with updated link costs. Perform subroutine Nestload, yielding auxiliary link flows $\tilde{\mathbf{y}}^{\mathbf{n}}$.
4. Move:

$$\mathbf{y}^{\mathbf{n}} = \mathbf{y}^{\mathbf{n-1}} + \frac{\tilde{\mathbf{y}}^{\mathbf{n}} - \mathbf{y}^{\mathbf{n}}}{n}$$

5. Convergence check: Perform a convergence check. If convergence is attained, finish; otherwise, go back to Step 2.

The core of the algorithm is the subroutine Nestload, which assigns the flows according to the link-nest model. Route loading for each origin-destination pair *pq* is described separately to facilitate understanding of the rationale behind the link-nested route choice model. The subroutine Nestload passes through three main stages:

1. For each O-D pair *pq* a set of relevant links ($a = 1 \ldots A$) is defined. This stage is implemented by passing through links sequentially without retaining the routes in memory.
2. For a given O-D pair *pq* and for each link-nest *a*, the *a*-shortest route and its cumulative relative loading (a weight calculated for all the links *b* in the route) is computed by the formula:

$$Load_b(a) = Load_b(a-1) + \sum_{r \in R(a)} \delta_{br} P(r|a) e^{\ln \alpha_{ar}^{pq} - c_r^{pq}} \tag{37}$$

It will be recalled that the conditional probability is given by Formula 8, which yields 1 for the maximum degree of nesting, apart from the case in which there are several equal *a*-shortest routes. In this case the conditional probability is reciprocal to the number of *a*-shorted routes. In parallel, a summation of total weight of all loadings for O-D pair *pq* is performed:

$$Load(a) = Load(a-1) + e^{\ln \alpha_{ar(a)}^{pq} - c_{ar(a)}^{pq}} \tag{38}$$

3. After the set of link-nests is exhausted, a single normalization is performed for all links. Link loading corresponding to the demand for origin-destination pair *pq* is computed by the formula:

$$y_b^{pq} = \frac{Load_b(A)}{Load(A)} d^{pq} \tag{39}$$

The subroutine Nestload works according to the following algorithm:

- For each origin *p*,
- For each destination *q*,
- For each nest (link) *a*,
- If link *a* is relevant, then,
- Build *a-shortest path,*
- Compute the loading share of each link *b* in the nest *a* by Formula 37,
- Accumulate the total weight of all loadings by Formula 38, and
- Assign the demand to each link *b* by Formula 39.

The relevance of link *a* with respect to the O-D pair *pq* is based on comparison of the length of the cheapest path through link *a* to the length of the shortest path between *p* and *q* (not necessarily passing through link *a*). It is clear that this ratio will always be greater or equal to 1. For practical purposes a threshold ratio of 1.5 to 3.0 is employed to eliminate overlong routes. If the threshold is high (10, for example), then almost all links will be relevant.

Building the *a*-shortest path constitutes a technical problem because of the presence of a nonadditive logarithmic term along with the standard additive term. Indeed, taking into account the relationship of Equation 11, $\max(\ln \alpha_{ar}^{pq} - c_r^{pq})$ is reached on the same route as is $\min(\ln L_r^{pq} + c_r^{pq})$ across routes passing through link *a*. In this research a special subroutine was developed based on shortest path deviation analysis. Deviation is defined at the O-D level as a situation in which the shortest paths defined by these two terms are different.

The shortest path deviation was introduced into the program to confine the search for the shortest log-path (which is very time consuming) to a small number of O-D pairs in which deviation occurs. If there is no deviation, the simple shortest path procedure is employed.

The all-pairs shortest path is a well-known problem. In this case, for each origin the tree-building algorithm based on the d'Esopo-Pape method is applied and the predecessor node matrix is stored for path restoration in the loading subroutine.

The proposed algorithm is appropriate for small networks. The core of the problem is that the basic procedure is implemented separately for each O-D pair. This in turn stems from the complexity of building the *a*-shortest path with a nonadditive term.

A possible means to significantly speed the process is to use a single criterion (congestion time) in both terms. In this case $\max(\ln \alpha_{ar}^{pq} - c_r^{pq})$ is reached on the same route as is $\min(c_r^{pq})$ across routes passing through link *a*. This makes it possible to simultaneously handle all destinations in conjunction with one origin. However, this leads to a more complicated structure of the route choice model in which the inclusion coefficients defined by Equation 11 are variable.

EMPIRICAL ANALYSIS

For the purposes of illustration within the limited framework of this report, the simplified network used by Cascetta et al. (*9*) was chosen. The network structure consists of 11 links and is presented in Figure 3, which shows free-flow generalized costs. No scaling is necessary in the route choice model because link performance is initially represented in generalized cost units.

Only one O-D pair is considered. There are five possible routes, two of them (L1,2,3,4 and L8,9,10,11) are disjoint and the others (L1,5,9,10,11 and L1,2,6,10, 11 and L1,2,3,7,11) are overlapping. The ordinary BPR function has been used for congestion cost calculation.

Two opposing network states are analyzed: uncongested and congested. In both states demand between the origin and the destination is specified as 100 trips. In the uncongested state the capacity of each link is specified as 1,000, so that the link costs are practically at the free-flow level. In the congested state the capacity of each link is set to 50, meaning that the specified demand corresponds to the maximum possible flow according to overall network capacity.

By using these two network states, the performance of three assignment models are compared:

- Deterministic user equilibrium,
- Stochastic user equilibrium by multinomial logit route choice model with Dial's loading procedure (Equation *5*), and
- Stochastic user equilibrium by the link-nested logit model of route choice with the loading procedure described above.

Assignment results are depicted in Figure 3 at the link level. Several conclusions can be drawn. For the uncongested state,

1. The deterministic model yields a similar probability for disjoint routes but almost ignores the overlapping routes. In particular, an absolute zero flow on link 6 seems unreasonable.
2. The multinomial logit model yields an unrealistic flow pattern with an equal distribution of flows (20 percent) among all five routes without taking into account route overlap.

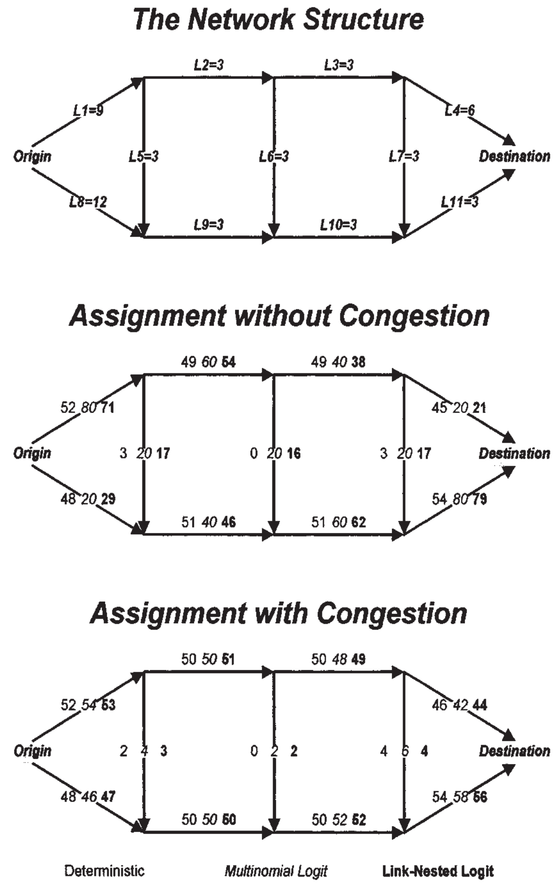

FIGURE 3 Comparison of assignment model performance.

3. The link-nested logit model yields a reasonable result, of the two disjoint routes together attract 50 percent of the flow, while the three overlapping routes obtain 16 to 17 percent each, leading to a link 1 flow of 71 percent against 29 percent for link 8. This reflects the number of route choice opportunities related to a choice of link 1. It is worth noting that all three overlapping routes obtain almost the same loading because all of them have the same degree of overlap with disjoint routes, although the relative location of common links are different.

For the congested state, the performances of all three models are virtually indistinguishable. This has been mentioned by other authors (*14*) and means that the route choice issue is important primarily for uncongested network fragments. The unreasonable zero flow on the link 6 in the deterministic assignment again is obtained, stemming from the fact that the route L1,2,6,10,11 passing through link 6 includes links 1,2,10, and 11, which are predetermined by the network structure to be the most congested.

CONCLUSIONS

1. Probabilistic route choice modeling faces a problem of proper treatment of overlapping routes that cannot be resolved by the multinomial logit model. Previously proposed alternative modeling constructs were based either on nonlogit choice models (probit, linear) or on the logit extensions (C-logit). All of them require either explicit route enumeration or a simulation technique at the link level for stochastic network loading.

2. The link-nested logit model can be used for route choice modeling. This model is derived from the GEV class and has a closed, analytically convenient, and behaviorally tractable form. The link-nested model has a flexible correlation structure which allows modeling of traveler choice among various routes with a differential degree of overlap.

3. The link-nested logit model of route choice can be incorporated into a stochastic user equilibrium framework. One possibility is a generalization of the Fisk's logit-based stochastic assignment that leads to a convex mathematical program with respect to route loading variables.

4. The special properties of the link-nested logit model in the particular case of a maximum degree of nesting, allow for stochastic network loading without explicit route enumeration.

5. Experimental application in a small network shows the clear advantage of the link-nested logit model over the deterministic and multinomial logit models with respect to trip loading quality. However, the computational efficiency of the loading procedure still is hampered by the multiple repetition of the *a*-shortest path searching for each O-D pair.

6. Further research is necessary to improve the efficiency of the assignment algorithm with respect to both stochastic loading and convergence to a network equilibrium.

7. After improving its computational efficiency, a comparative analysis of the link-nested logit model to alternative assignment models and to traffic counts in real-world networks would be welcome.

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