Paper No. 01-2679

Stochastic User Equilibrium Formulation for Generalized Nested Logit Model

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The route choice problem is complicated in typical transportation networks because of the size of the choice set and because of the overlapping problem since many routes share links. Well-known models like the probit and the logit were further developed in an attempt to overcome these problems. The logit model has the appeal of being relatively close to the probit model while keeping a convenient analytical closed form. However, the simple multinomial logit model cannot correctly represent route choice, especially with respect to the overlapping problem. Other hierarchical logit models can potentially overcome the overlapping problem. The recently developed generalized nested logit (GNL) model is found to be very suitable for route choice, as is the cross-nested logit (CNL) model. The inclusion of the congestion effect in the route choice problem is accounted for in stochastic user equilibrium (SUE) problems. The development of a SUE formulation for the GNL model is presented. In addition, how to adapt the GNL model to route choice in a way similar to that of the CNL model is shown. An equivalent SUE formulation for the GNL model is developed. In this way, a unified framework is presented to relate GNL-type models, which are derived from discrete choice theory, with aggregate entropy formulations. A preliminary algorithm is developed to illustrate the potential application of the GNL formulation for real networks.

In the last several years, there has been a renewed interest in using discrete choice theory to model route choice. The problem is complicated in typical transportation networks because of the size of the choice set (the alternative routes) and also because of the overlapping problem since many routes share links in common. Well-known models like the probit and the logit were further developed in the literature in an attempt to overcome these problems.

Yai et al. (*1*) considered the probit model with a structured covariance matrix. The authors related the covariance terms of the probit model to the ratio of length of the common part of the routes divided by the total route length. Maher and Hughes (*2*) also considered the probit model in developing a stochastic network loading method that avoids route enumeration. However, in both studies the probability calculation demands either some simulation method or analytical approximation.

The logit model has the appeal of being relatively close to the probit model while keeping a convenient analytical closed form. However, the simple multinomial logit (MNL) model cannot correctly represent route choice, especially with respect to the overlapping problem. Cascetta et al. (*3,* pp. 697–711) developed the C-logit model, which can also be interpreted as an implicit availability (or perception) model. Prashker and Bekhor (*4*) adapted both the cross-nested logit (CNL) model of Vovsha (*5*) and the paired combinatorial logit (PCL) model of Chu (*6,* pp. 295–309) to route choice situations.

The inclusion of the congestion effect in the route choice problem is accounted for in stochastic user equilibrium (SUE) problems. There are two basic different optimization formulations for the SUE problem. The first formulation is that due to Sheffi and Powell (*7*), which can include a more general class of models, and the second formulation is based on entropy terms, which is specific to the logit class of models. Fisk (*8*) formulated the first equilibrium formulation specific for the MNL model, and Bekhor and Prashker (*9,* pp. 351–372) developed equilibrium formulations for the CNL and PCL models.

Recently, Wen and Koppelman (*10*) presented a generalized nested logit (GNL) model, which stemmed from the generalized extreme value (GEV) theorem of McFadden (*11,* pp. 75–96). The GNL model includes the PCL and the CNL models as special cases. Papola (*12*) presented developments of the CNL model, which yields to the GNL model. Moreover, he developed a structured covariance matrix for the CNL model and related it to different nesting possibilities. This interesting approach is discussed later in this paper.

The development of a SUE formulation for the GNL model is presented. In addition, it is shown how to adapt the GNL model to route choice in a similar way to that of the CNL model. An equivalent SUE formulation for the GNL model is developed, and a preliminary algorithm is presented. In this way, a unified framework is presented to relate GNL-type models, which are derived from discrete choice theory, and aggregate entropy formulations.

In the next section the basic characteristics of the GNL model are presented followed by a new SUE formulation that yields the GNL model as the solution for the mathematical problem. The adaptation of the general GNL model to the route choice situation is considered next. Special attention is paid to the nesting coefficient parameter. A preliminary path-based algorithm to solve the SUE problem with the GNL formulation is presented, and the findings of this paper are summarized.

GNL MODEL

The GNL model following Wen and Koppelman (*10*) is developed from the GEV theory as follows. Consider the following function:

$$G(y_1, y_2, \ldots, y_K) = \sum_m \left[\sum_{k \in K_m} (\alpha_{km} y_k)^{\frac{1}{\mu_m}} \right]^{\mu_m} \tag{1}$$

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where

K_m = set of all alternatives included in nest *m*,
α_{km} = inclusion coefficient of alternative *k* in nest *m*,
μ_m = nesting coefficient (specific for each nest *m*), and
y_k characterizes the value for each alternative.

The inclusion coefficient characterizes the portion of alternative *k* assigned to nest *m*; α_{km} must satisfy the following conditions:

$$\sum_m \alpha_{km} = 1 \quad \forall k \in K_m \qquad \alpha_{km} \geq 0 \tag{2}$$

In addition, the nesting coefficient must satisfy the condition $0 < \mu_m \leq 1$ for consistency with random utility maximization.

The function represented in Equation 1 satisfies the conditions required in the GEV theorem as follows:

- $G(\ldots)$ is nonnegative [it is assumed that $y_k = \exp(V_k)$, where V_k is the utility of alternative *k*];
- $G(\ldots)$ is homogeneous of degree 1;
- $\lim_{y_k \to \infty} G(\ldots) = \infty$ for each *k;* and
- $G(\ldots)$ has *k*th partial derivatives, which are nonnegative for odd *k* and nonpositive for even *k.*

If the GEV theorem is applied, the probability of choosing an alternative is given by the following expression:

$$P(k) = \frac{\sum_m \left([\alpha_{km} \exp(V_k)]^{\frac{1}{\mu_m}} \left\{ \sum_{k \in K_m} [\alpha_{km} \exp(V_k)]^{\frac{1}{\mu_m}} \right\}^{\mu_m - 1} \right)}{\sum_m \left\{ \sum_{k \in K_m} [\alpha_{km} \exp(V_k)]^{\frac{1}{\mu_m}} \right\}^{\mu_m}} \tag{3}$$

Equation 3 can be decomposed into marginal and conditional probabilities as follows:

$$P(k) = \sum_m P(m) P(k|m) \tag{4}$$

where $P(m)$ is the marginal probability of choosing nest *m*, given by the following expression:

$$P(m) = \frac{\left\{ \sum_{k \in K_m} [\alpha_{km} \exp(V_k)]^{\frac{1}{\mu_m}} \right\}^{\mu_m}}{\sum_m \left\{ \sum_{k \in K_m} [\alpha_m \exp(V_k)]^{\frac{1}{\mu_m}} \right\}^{\mu_m}} \tag{5}$$

and $P(k|m)$ is the conditional probability of choosing alternative *k* if nest *m* is selected given by the following expression:

$$P(k|m) = \frac{[\alpha_{km} \exp(V_k)]^{\frac{1}{\mu_m}}}{\sum_{k \in K_m} [\alpha_{km} \exp(V_k)]^{\frac{1}{\mu_m}}} \tag{6}$$

Wen and Koppelman (*10*) showed that many other GEV-type models could be derived from the GNL model. In particular, the CNL model was obtained as a straightforward restriction of the GNL model with respect to the nesting coefficient. The CNL model assumes that all μ_m parameters in the GNL model are equal. Papola (*12*) derived the CNL model by assuming that the nesting coefficients may differ, as in the GNL model. The resulting model is then identical to the GNL model.

After this brief presentation of the GNL model as derived from discrete choice theory, a mathematical formulation is presented that gives as its solution the GNL model.

GNL-EQUIVALENT MATHEMATICAL FORMULATION

A mathematical formulation is given for which the solution obtained is the GNL model. The GNL model is a hierarchical choice model that can be decomposed into marginal and conditional probabilities. Similarly, the objective function has to be decomposed into two entropy terms instead of only one in the MNL model. In this way, it is possible to obtain the conditional and marginal probabilities as solutions for the equivalent formulation.

Consider the following mathematical formulation:

$$\min Z = Z_1 + Z_2 + Z_3$$

$$Z_1 = \sum_a \int_0^{x_a} c_a(w)\,dw$$

$$Z_2 = \frac{1}{\theta} \sum_{rs} \sum_m \sum_k \mu_m f_{mk}^{rs} \ln \frac{f_{mk}^{rs}}{(\alpha_{mk}^{rs})^{1/\mu_m}} \tag{7}$$

$$Z_3 = \frac{1}{\theta} \sum_{rs} \sum_m (1 - \mu_m) \left(\sum_k f_{mk}^{rs} \right) \ln \left(\sum_k f_{mk}^{rs} \right)$$

subject to

$$\sum_m \sum_k f_{mk}^{rs} = q^{rs} \quad \forall r, s$$

$$f_{mk}^{rs} \geq 0 \quad \forall m, k, r, s$$

where

f_{mk}^{rs} = flow on path *k* of nest *m* between *r* and *s* (origin and destination, respectively),
q^{rs} = demand between *r* and *s,*
c_a = cost on link *a,*
x_a = flow on link *a,*
α_{mk}^{rs} = inclusion coefficient of path *k* in nest *m* between *r* and *s,*
θ = dispersion coefficient,
μ_m = nesting coefficient, and
$f_{mk}^{rs} \ln [f_{mk}^{rs}/(\alpha_{mk}^{rs})^{1/\mu}]$ is defined as zero for either $f_{mk}^{rs} = 0$ or $\alpha_{mk}^{rs} = 0$.

There are two main differences between the mathematical formulation presented above and Fisk's (*8*) equivalent formulation for the MNL model: first, the inclusion of another entropy term (Z_3), corresponding to the higher choice level, and second, the modification of the entropy term (Z_2) to include the inclusion coefficient. The summation of the path flows is decomposed by the *m* links (nests).

The foregoing formulation is very similar to the equilibrium formulation for the CNL model presented by Bekhor and Prashker (*9*, pp. 351–372). The first-order conditions for the solution of the foregoing optimization problem correspond to the GNL model. Only the main steps of the proof are presented here. A detailed proof of existence and uniqueness of the problem can be found in the discussion by Bekhor and Prashker.

Existence Conditions

The first-order conditions for a solution of the problem are obtained by forming the Lagrangian function for the formulation just presented. The indexes r and s are omitted for simplicity of notation:

$$L = Z_1 + Z_2 + Z_3 + \lambda\left(q - \sum_m \sum_k f_{mk}\right) \quad (8)$$

where λ is the Lagrange coefficient. The partial derivatives are obtained as follows:

$$\frac{\partial Z_1}{\partial f_{mk}} = \sum_a \frac{\partial Z_1}{\partial x_a}\frac{\partial x_a}{\partial f_{mk}} = \sum_a c_a \delta_{ak} = c_k \quad (9)$$

$$\frac{\partial Z_2}{\partial f_{mk}} = \frac{\mu_m}{\theta}\ln\frac{f_{mk}}{(\alpha_{mk})^{1/\mu_m}} + \frac{\mu_m}{\theta} \quad (10)$$

$$\frac{\partial Z_3}{\partial f_{mk}} = \frac{1-\mu_m}{\theta}\ln\left(\sum_k f_{mk}\right) + \frac{1-\mu_m}{\theta} \quad (11)$$

Equating the partial derivatives to zero and after some manipulations, the following expression is obtained:

$$(f_{mk})\left(\sum_k f_{mk}\right)^{\frac{1-\mu_m}{\mu_m}} = \exp[(\theta\lambda - 1)/\mu_m](\alpha_{mk})^{1/\mu_m} \times \exp[(-\theta c_k)/\mu_m] \quad (12)$$

Summing the above expression by route k provides the following expression:

$$\left(\sum_k f_{mk}\right)^{\frac{1}{\mu_m}} = \exp[(\theta\lambda - 1)/\mu_m]\sum_k (\alpha_{mk})^{1/\mu_m} \times \exp[(-\theta c_k)/\mu_m] \quad (13)$$

Elevating both sides to μ_m,

$$\sum_k f_{mk} = \exp(\theta\lambda - 1)\left\{\sum_k (\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu]\right\}^{\mu} \quad (14)$$

Summing the above expression by link (nest) m,

$$\sum_m \sum_k f_{mk} = q = [\exp(\theta\lambda - 1)]\sum_m \left\{\sum_k (\alpha_{mk})^{1/\mu}\exp[(-\theta c_k)/\mu]\right\}^{\mu} \quad (15)$$

Finally dividing Equation 14 by Equation 15 leads to

$$P(m) = \frac{\sum_k f_{mk}}{q} = \frac{\left\{\sum_k (\alpha_{mk})^{1/\mu_m}\exp[(-\theta c_k)/\mu_m]\right\}^{\mu_m}}{\sum_m \left\{\sum_k (\alpha_{mk})^{1/\mu_m}\exp[(-\theta c_k)/\mu_m]\right\}^{\mu_m}} \quad (16)$$

which corresponds to the marginal probability that nest m will be chosen. The conditional probability is obtained by dividing Equation 13 by Equation 14, which yields the conditional probability that route k will be chosen in nest m:

$$P(k|m) = \frac{f_{mk}}{\sum_k f_{mk}} = \frac{(\alpha_{mk})^{1/\mu_m}\exp(-\theta c_k/\mu_m)}{\sum_k (\alpha_{mk})^{1/\mu_m}\exp(-\theta c_k/\mu_m)} \quad (17)$$

Equations 16 and 17, respectively, correspond to Equations 5 and 6, assuming that the deterministic utility component V_k in Equations 5 and 6 is represented by ($-\theta c_k$) in Equations 16 and 17. In this way, the mathematical formulation presented in Equation 7 corresponds to a SUE formulation, for which the path-flow solution is obtained according to the GNL model.

Uniqueness Conditions

The feasible region and Z_1 are the same as Fisk's formulation, and therefore are convex. It is left to show that the components Z_2 and Z_3 are convex. Twice differentiating both expressions for a path-flow variable gives the following:

$$\frac{\partial^2 Z_2}{\partial f_{mk}\partial f_{ml}} = \begin{cases} \dfrac{\mu_m}{\theta f_{mk}} & l = k \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

$$\frac{\partial^2 Z_3}{\partial f_{mk}\partial f_{ml}} = \frac{1-\mu_m}{\theta \sum_k f_{mk}} \quad (19)$$

The Hessian matrix of Z_2 is positive definite, and the Hessian matrix of Z_3 is positive semidefinite (all elements in the matrix are equal to one another, and the determinant is equal to zero). This ensures convexity of the whole objective function, and hence the solution is unique in terms of the path-flow variable f_{mk}.

Collapsing GNL to MNL

The GNL model collapses to the MNL model for the case $\mu_m = 1$. In the following it is shown that the above equilibrium formulation collapses to the MNL equilibrium formulation for the case $\mu_m = 1$. In the formulation of the problem (Equation 7), the term Z_3 vanishes in this case. However, the term Z_2 does not immediately turn to the equivalent MNL formulation. Only when the first-order conditions are developed is the MNL model achieved. The following development shows the proof for this case.

The partial derivatives are obtained as follows:

$$\frac{\partial Z_1}{\partial f_{mk}} = \sum_a \frac{\partial Z_1}{\partial x_a}\frac{\partial x_a}{\partial f_{mk}} = \sum_a c_a \delta_{ak} = c_k \quad (20)$$

$$\frac{\partial Z_2}{\partial f_{mk}} = \frac{\mu_m}{\theta}\ln\frac{f_{mk}}{(a_{mk})^{1/\mu_m}} + \frac{\mu_m}{\theta} = \frac{1}{\theta}\ln\frac{f_{mk}}{\alpha_{mk}} + \frac{1}{\theta} \quad (21)$$

Equating the partial derivatives to zero and multiplying by θ gives the following:

$$\theta c_k + \ln\frac{f_{mk}}{\alpha_{mk}} + 1 - \theta\lambda = 0 \quad (22)$$

where λ is the Lagrange coefficient. Rearranging terms and taking the exponent,

$$f_{mk} = \alpha_{mk}\exp(\theta\lambda - \theta c_k - 1) \quad (23)$$

Summing the flows for all nests m gives the following:

$$\sum_m f_{mk} = \exp(\theta\lambda - \theta c_k - 1)\sum_m \alpha_{mk} \quad (24)$$

If one recalls the regularity constraint imposed on the inclusion coefficients ($\Sigma_m \alpha_{mk} = 1$), the following expression results:

$$\sum_m f_{mk} = f_k = \exp(\theta\lambda - \theta c_k - 1) \quad (25)$$

Summing the above expression for all routes k gives the following expression:

$$\sum_k f_k = \exp(\theta\lambda - 1)\sum_k \exp(-\theta c_k) \quad (26)$$

Finally, dividing Expression 25 by Expression 26 gives the following:

$$\frac{f_k}{\sum_k f_k} = \frac{\exp(-\theta c_k)}{\sum_k \exp(-\theta c_k)} \quad (27)$$

The above expression corresponds to the MNL model.

This concludes the presentation of a new equivalent mathematical formulation for the stochastic user equilibrium problem, which corresponds to the GNL model. Next the adaptation of the parameters of the GNL model to route choice is discussed.

ADAPTING GNL PARAMETERS TO ROUTE CHOICE SITUATION

The GNL model has two sets of parameters, which have the following interpretation according to Wen and Koppelman (*10*): the inclusion coefficient α_{mk}, which indicates the portion of alternative k assigned to nest m, and the nesting coefficient μ_m, which indicates the logsum (or dissimilarity parameter) for nest m. It is assumed here that the inclusion coefficients can be related to network topology.

Inclusion Coefficients

Both sets of coefficients are related to the network topology in a fashion similar to that of Prashker and Bekhor (*4*). The inclusion coefficient α_{mk} can be expressed as follows:

$$\alpha_{mk} = \left(\frac{L_m}{L_k}\right)^{\gamma} \delta_{mk} \quad (28)$$

where

- L_m = link length,
- L_k = path length,
- δ_{mk} = 1 if link m is on route k and 0 otherwise, and
- γ = parameter to be calibrated, which reflects the driver's perception of similarity among routes (in this research, it is assumed for simplicity that γ is equal to 1).

The inclusion coefficient defined in Equation 28 is dependent only on the network topology. The derivation of the GNL model from the GEV class requires the inclusion coefficients to be independent of the generalized cost. If the inclusion coefficient is assumed to be proportional to the link costs (instead of link lengths), α_{mk} is also dependent on congestion. However, this assumption leads to a more complex analytical construct, not only for the route choice function but also for the equilibrium formulations. Therefore, in this study it is assumed that this coefficient is not dependent on congestion.

The formulation of the GNL model presented earlier permits an alternative (in this case, a route) to belong to more than one nest (in this case, a link). The crossing effect is represented by the inclusion coefficient α_{mk}, $0 \le \alpha_{mk} \le 1$. The nested logit model is a special case of the GNL model in which the coefficient α_{mk} is either zero or 1. By assigning only binary values to α_{mk}, an alternative can only belong to one nest, as in the nested logit model.

Nesting Coefficients

The difference between the GNL and the CNL models is the nesting coefficient μ_m. The CNL model assumes that all nests have the same nesting coefficient μ. The relation of the nesting coefficient to the route choice situation was treated in different ways. Vovsha and Bekhor (*13*) assumed that $\mu \to 0$, which yields the link-nested logit model. Papola (*12*) showed a general expression of the variance-covariance matrix of the CNL model. He showed that the link-nested logit model with the assumption of $\mu \to 0$ is very suitable for the route choice situation. However, this extreme case is suitable only when the total route costs are equal, as discussed later in a separate section.

The more general structure of the GNL model allows for assigning different nesting coefficients to different nests. It is interesting to note that the nesting coefficient captures the similarity between nests, whereas the inclusion coefficient α_{mk} captures the similarity within a nest. Different route choice models can then be obtained by suitably representing the similarity among routes. For example, the PCL model can be derived from the GNL model assuming that all allocation (inclusion) coefficients α_{mk} are equal, and the nesting coefficients μ_{kj} are arranged so that they represent the similarity or dissimilarity between a pair of alternatives k and j. Hence, the similarity among routes is captured by the nesting coefficient as follows:

$$\mu_{kj} = 1 - \left[\frac{L_{kj}}{(L_k L_j)^{0.5}}\right] \quad (29)$$

where L_{kj} is the length of the common part of routes k and j. Equation 29 confines the similarity-index boundaries between zero and 1. These conditions have to hold for the PCL model to be consistent with random utility maximization. If μ_{kj} approaches zero, this condition indicates that all the links of a path are completely equal to the links of the other path (maximum overlap). On the other hand, if μ_{kj} is equal to 1, the two paths have no link in common. In this case, the PCL model is identical to the MNL model.

The nesting coefficients for the GNL model can also be directly related to the network topology. The similarity between nests is inversely related to the nesting coefficient. In this way, it is possible to obtain the following expression for the nesting coefficients:

$$\mu_m = 1 - \frac{1}{N_m}\sum_k \alpha_{mk} \quad (30)$$

where N_m is the number of routes passing through link (nest) m. In this way, the nesting coefficient is also directly obtained from the network topology by calculating an average value of the inclusion coefficients.

Ben-Akiva et al. (*14*, pp. 299–330) investigated route choice behavior in interurban networks. In their work, the nested logit model was considered, and different criteria (labels) were considered for route choice, such as the shortest route, the quickest route, maximizing motorway use, and so on. The nesting coefficient was then calibrated with survey data. Relating the parameters of the GNL model only to network topology, as presented here, appears too restrictive. Further research is needed to include behavior estimates in the GNL route choice model.

Special Case of $\mu \to 0$

The GNL and CNL models reduce to a simple functional form when $\mu \to 0$ as follows. The starting point is the marginal probability of choosing a nest, given by the following expression:

$$P(m) = \frac{\left\{\sum_k [\alpha_{mk} \exp(-c_k)]^{1/\mu}\right\}^{\mu}}{\sum_b \left\{\sum_k [\alpha_{bk} \exp(-c_k)]^{1/\mu}\right\}^{\mu}} = \frac{\exp(-\tilde{c}_m)}{\sum_b \exp(-\tilde{c}_b)} \tag{31}$$

where $\tilde{c}_m$ is the component cost of nest m, defined by the following expression:

$$\tilde{c}_m = \ln\left[\sum_m \exp\left(\frac{\ln \alpha_{mk} - c_k}{\mu}\right)\right]^{\mu} \tag{32}$$

As $\mu \to 0$, the expression inside the exponent tends to infinity, indicating that the maximum value of $(\ln \alpha_{mk} - c_k)$ for a given route k will be much larger than that for other routes. Therefore, it is possible to substitute the maximum value for the sum before the exponent, as follows:

$$\lim_{\mu \to 0} \ln\left[\sum_m \exp\left(\frac{\ln \alpha_{mk} - c_k}{\mu}\right)\right]^{\mu} = \mu \ln\left[\exp \max_k\left(\frac{\ln \alpha_{mk} - c_k}{\mu}\right)\right]$$
$$= \max_k(\ln \alpha_{mk} - c_k) \tag{33}$$

The marginal probability of choosing a nest is equal to the ratio between the exponent of the composite costs. In this special case, the probability will be dependent on the maximum value of $(\ln \alpha_{mk} - c_k)$, which is obtained at the shortest path passing through link (or nest) m.

The marginal probability of choosing a nest when $\mu \to 0$ reduces to

$$\lim_{\mu \to 0} P(m) = \frac{\exp[\ln \alpha_{mk(m)} - c_{k(m)}]}{\sum_b \exp[\ln \alpha_{mk(b)} - c_{k(b)}]} \tag{34}$$

where $k(m)$ is the best route of an origin-destination (O-D) pair passing through link m (shortest m-path).

The conditional probability of choosing route k with a given link m is simply

$$\lim_{\mu \to 0} P(k|m) = \begin{cases} \dfrac{1}{R(m)} & k \in R(m) \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

where $|R(m)|$ is the number of equal shortest m-paths.

Since there is only a small probability of having equal shortest m-paths in a real network, the conditional probability can be omitted, and thus the probability of choosing a route equals the marginal probability of choosing a nest.

A simple example is presented here to show that the assumption of $\mu \to 0$ may yield some counterintuitive results. Figure 1 shows the network for this example. The numbers on each link indicate travel costs. In this network, all link costs are constant and equal to 2 units except the intermediate link, in which the link length varies from 0 to 3. When a is equal to zero, the three routes have equal length. As a increases, the probability of using the intermediate link should decrease because the length of the Z-route becomes longer than the other two routes.

Figure 2 compares the results for different values of the nesting coefficient μ of the CNL and the GNL models. It should be recalled that when μ is 1, the CNL model collapses to the MNL model. The results presented in Figure 2 show that the CNL model produces counterintuitive results for $\mu \to 0$ in the range 0 to 0.5. The explanation for this phenomenon is given next.

In this network, when the link length on the intermediate link increases, there is no tie between the paths. However, the total route length of each path is approximately the same for small values of the intermediate link.

The shortest m-path passing through this link is always the same, and the composite cost equals

$$\ln \alpha_{mk} - c_k = \ln\left(\frac{L_a}{L_k}\right) - L_k \tag{36}$$

When the link length is small enough, the logarithm will yield large negative values, comparable with the total route length. Thus, for special cases in which a route (with common segments) has an exclusive link belonging to the route, the degenerated case will produce counterintuitive results.

PRELIMINARY ALGORITHM

The preliminary path-based algorithm is an adaptation of the disaggregate simplicial decomposition (DSD) algorithm developed by Damberg et al. (*15*). The differences are the direction calculations and step-size determination, which are computed according to the GNL equilibrium model. The algorithm is based on the fact that all stochastic equilibrium formulations have in common that the solution of the optimization problem is the probability function.

The algorithm is simplified by generating a subset of routes (up to 40 for each O-D pair) before the equilibrium iterations. The algorithm is described as follows:

1. Initialize: Find an initial feasible solution $\mathbf{f}^1$. Set $n = 1$.
2. Update: Assign the path flows $\mathbf{f}^n$ to yield link flows $\mathbf{x}^n$. Compute travel costs $\mathbf{c}(\mathbf{x}^n)$.

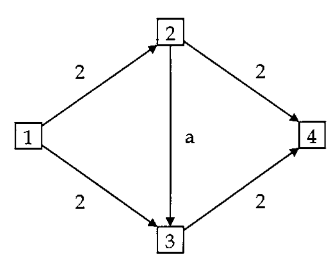

FIGURE 1 Three correlated routes.

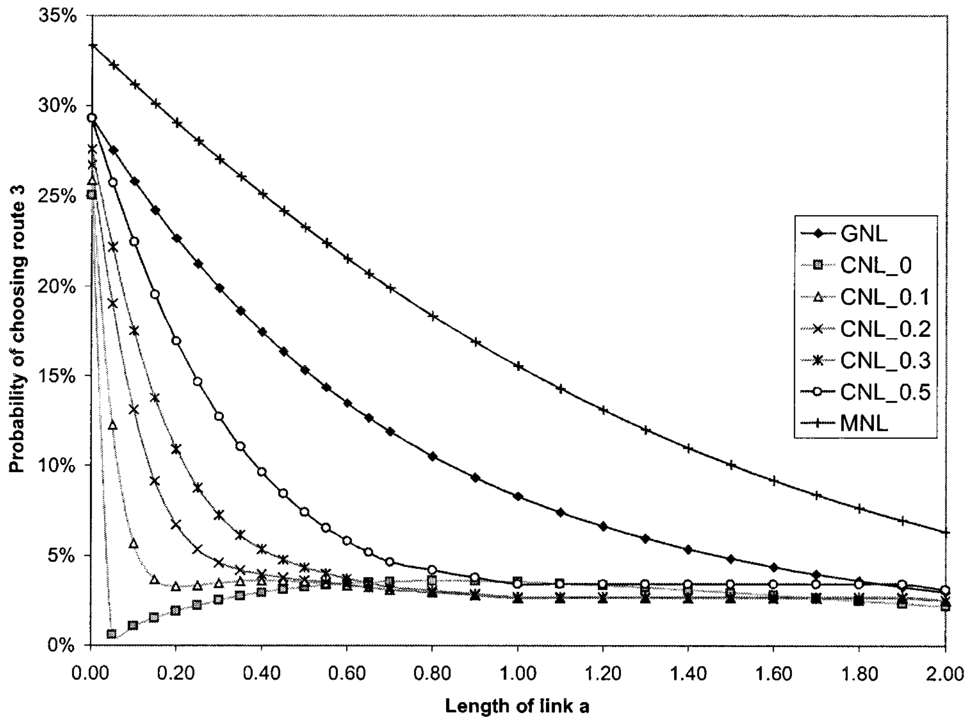

FIGURE 2 Influence of nesting coefficient.

3. Direction finding: Perform a stochastic network loading based on the current set of link travel costs. This loading yields an auxiliary solution $\mathbf{h}^n$.

4. Step-size determination: Perform a line search to find the optimal step size:

$$\lambda^{(n)} = \min_{\lambda \in [0,1]} Z\{\mathbf{f}^{(n)} + \lambda[\mathbf{h}^{(n)} - \mathbf{f}^{(n)}]\}$$

5. Move: Set $\mathbf{f}^{n+1} = [\mathbf{f}^n + \lambda (\mathbf{h}^n - \mathbf{f}^n)]$.

6. Convergence test: If $Z(x^n) - Z(x^{n+1}) \leq \epsilon$ (or another convergence criterion is met), stop. Otherwise, set $n = n + 1$ and go to Step 2.

The difference between this algorithm and a path-based method of successive averages (MSA) algorithm is that the step size is optimized using the GNL optimization formulation. The evaluation of the objective function is more complicated than with the MNL model because of the additional entropy term. Moreover, the stochastic network loading phase with the new formulations involves path comparisons to capture the similarity effect.

It should be noted that in the specific case in which the routes are generated a priori, it is possible to calculate the similarities between the paths before the assignment because the similarities defined in this paper depend only on the network topology. However, if a more general definition for the similarities were used (including perhaps the travel costs), the external calculation of the similarities would not be possible.

A real network was coded to test the algorithm with a limited number of routes. The data source is a simplified Haifa–Tel Aviv interurban network composed of 20 nodes, 56 links, and 83 O-D pairs with positive flow. The purpose of this test was to check the validity of the results for a real network. Although the network dimensions in this test are very modest, the number of paths for some O-D pairs is quite high. The demand matrix was inferred from traffic counts in the a.m. peak hour, and the total number of trips in the matrix is 18,350.

The results from the previous tests indicate that there is a tradeoff between the large number of iterations in the MSA and the complexity of the step-size determination in the modified Frank-Wolfe method. In this test, it is possible to get an idea of the price paid for using a better behavioral model compared with simpler models.

The assignments were conducted for the following input parameters:

- Dispersion parameter equal to 1,
- Nesting coefficient equal to 1 (does not affect the computation time and serves to validate the results with the MNL model),
- Maximum number of routes (for each O-D pair) equal to 35, and
- Convergence check equal to 0.1 (maximum 10 percent change on each link of the network).

The basis for the network performance is the logit assignment with the MSA algorithm. The ratio of the CPU time for the proposed algorithm with respect to the MSA is computed for a specified convergence value. All the assignments were performed using a computer program coded in a personal computer. Table 1 shows the results.

The results presented in Table 1 clearly indicate the superiority of the DSD algorithm with respect to the MSA algorithm. This conclusion was also verified by Damberg et al. (*15*) when they compared their algorithm with respect to the MSA. The difference between the algorithm of Damberg et al. and the algorithm in this paper is basically the step-size computation: their algorithm used the Armijo step-size rule, whereas here the step size was determined by minimizing the objective function.

TABLE 1 Algorithm Performance for Haifa–Tel Aviv Network

Route Choice Model	Algorithm	Time for 1 Iteration (seconds)	Number of Iterations to Converge	Total Time (seconds)	Ratio to MNL+MSA algorithm
MNL	MSA	2.3	140	322.0	1.00
GNL	MSA	8.8	140	1232.0	3.83
MNL	DSD	9.2	8	73.6	0.23
GNL	DSD	99.0	8	792.0	2.46

SUMMARY

It is demonstrated how the generalized nested logit (GNL) model could be adapted to the route choice situation. The approach given is to first present the model from discrete choice theory and to then develop a new mathematical formulation using aggregate entropy measures. The adaptation of the parameters of the GNL model to route choice is discussed, and a special case is analyzed.

The congestion effect is modeled by formulating an equivalent stochastic user equilibrium (SUE) problem, the solution of which yields the GNL model. It is shown that the aggregate formulation collapses to the multinomial logit (MNL) model when the nesting coefficient is equal to 1.

It is also indicated how to adapt the GNL model parameters. The inclusion coefficients are given a similar approach to that of the CNL model. The relation between the nesting coefficient and the inclusion coefficient is discussed. The GNL model can capture both the similarity within a nest (the allocation parameters) and the similarity among different nestings (the different nesting coefficients). The paired combinatorial logit (PCL) model, which is a special case of the GNL model, is presented to show the trade-off between the different similarities. It is suggested that the GNL parameters be obtained by some calibration procedure.

The special case $\mu \rightarrow 0$ is shown to be suitable for route choice only when the total route costs are equal, which is rarely the case in real networks. A simple example is presented to show that this case may produce counterintuitive results.

A preliminary algorithm was presented to illustrate the potential application of the GNL for real networks. This paper focused on the trade-off between the more accurate step size and the simpler method of successive averages (MSA) algorithm. Further investigation of the performance of the disaggregate simplicial decomposition (DSD) algorithm and investigation of alternative path-based algorithms is needed. However, the results presented in this paper indicate that the implementation of the GNL equilibrium formulation for a real network is feasible for large-scale networks.

ACKNOWLEDGMENTS

The helpful comments of Tomer Toledo (Ph.D. candidate at the Massachusetts Institute of Technology) and three anonymous reviewers are greatly appreciated.

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Publication of this paper sponsored by Committee on Traveler Behavior and Values.