The Factor of Revisited Path Size

Alternative Derivation

Piet H. L. Bovy, Shlomo Bekhor, and Carlo Giacomo Prato

The concept of path size attempts to capture correlations among routes in route choice modeling by including a correction term in the multinomial logit formulation. Several correction terms were proposed in the literature, yet no satisfactory derivation based on theoretical arguments is presented, raising doubts about the correct specification of the correction terms. This paper proposes the detailed and systematic derivation of a new formulation of the measure of path size and explicitly defines the assumptions involved in its derivation. The path size correction (PSC) factor results from the notion of aggregate alternative as well from the simplification of nested logit models. The new measure of path size offers a more natural interpretation of the correlation due to spatial overlap of alternative routes. Estimation of PSC-logit models in two real-world networks and calculation of predicted choice probabilities in synthetic networks allow comparison of the new path size measure with respect to the classic one. Estimates show similar performances between the models, and predictions illustrate better performances of the new version of the path size factor.

Route choice modeling has a problem with correlation among alternative routes in the choice set because of their partial overlap. On the one hand, this situation precludes adoption of the simple multinomial logit (MNL) model because of its inability to handle correlation among alternatives. On the other hand, models such as paired combinatorial logit (PCL), cross-nested logit (CNL), logit kernel, and probit can handle correlation to a certain extent but at the cost of much greater complexity and computational burden.

For practical applications, approximate procedures have been proposed such as C-logit (*1*) and path size logit (PSL) (*2*). These models try to capture the correlation among the alternatives by adding a correction term to the utility function of the MNL model to correct the calculated choice probabilities. Although these models are heuristic approximations in handling correlations, they have the advantage of adopting the well-known and simple logit structure and outperforming the MNL model.

Generally, C-logit and PSL present the problem that the adopted correction is a heuristic concept missing a well-founded derivation. Assumptions implicitly used are not known and no satisfactory derivation based on theoretical arguments is presented in the literature, thus raising doubts about the correct specification of the correction terms.

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This paper intends to fill that gap. Notwithstanding that C-logit and PSL are related concepts, the PSL approach is assumed as a departure point because of its more convincing logic (*3*). This paper proposes the derivation of a new path size expression different from the classic formulation, the significant extension of the theoretical derivation given for the classic formulation, and the clear identification of the assumptions needed for its derivation. The new and the classic expressions are compared by estimating route choice models for two real-world networks and by calculating predicted choice probabilities for a synthetic network.

Although the measures are mathematically different, the differences in estimation and prediction outcomes are small. Therefore, the real contribution is the sound path size derivation and the clarity about the assumptions made in including a path size measure in the utility function.

The paper is organized as follows. The next section discusses the correlation among alternative routes. The third section shows the derivation of the path size measure. The fourth section introduces the new derivation of the path size correction factor, and the fifth section presents an interpretation of this correction factor. The last section presents model estimation results for two real-world networks and compares predictions by using a synthetic route data set.

CORRELATION BETWEEN ALTERNATIVE ROUTES

Routes in surface transport networks mostly show a high degree of overlap with many other routes in the choice set. The spatial overlap leads to correlation between alternative routes because of the common utilities resulting from the overlapping parts. Because these correlations significantly influence the choice probabilities, several route choice model structures with different degrees of complexity have been proposed and successfully adopted to account for the correlation (*3*).

This paper is confined to the simplest models, such as C-logit and PSL, which account for the correlation by adding a specific term to the utility function of the MNL model. This term reduces the utility of an alternative route depending on the level of overlap with other routes in the choice set. For this reason, expressions for the correlation due to spatial overlap need to be specified.

In random utility modeling, alternative routes are assumed to have random utility distributions with variances and covariances. It is generally agreed that the random error variance in the case of routes somehow is related to some length or size measure of the route; see Daganzo and Sheffi (*4*) for a sound motivation. Alternatively, the length or size measure may be the number of constituting links, the length of the route in distance or time, or even the disutility of the route.

A general expression for the route error term variance σ that encompasses various specifications found in the literature (*4*) is as follows:

$$\sigma(\epsilon) = C \cdot L^m \tag{1}$$

where

ϵ = error term,
C = proportionality factor,
L = measure for route length or size, and
m = positive coefficient.

In common practice, m is set equal to 0.5, 1, or 2. Although values for m other than 1 can make sense from a behavioral perspective, these values are not recommendable from a logical and computational point of view.

If Relationship 1 is true for a route, it should also be true for part of a route, even for each individual link. This argument requires additivity of variances that is given only if m is equal to 1, as adopted throughout this paper. The additivity of variances implies the assumption of statistical independence of error terms among route parts and thus also among links of a route. Although there are good reasons for positive and negative error term correlations among route parts (and links), they are generally disregarded. At best, it is possible to assume that these correlations among links cancel out at the route level.

Covariance among different routes in a choice set can have various causes. At least one of them is the physical overlap among the routes. If Relationship 1 is assumed, it logically follows that the covariance due to overlap is related to the length or size of the common route parts. Thus, for any pair of routes i and j,

$$\sigma(\epsilon_i, \epsilon_j) = C \cdot l_{ij} \tag{2}$$

where l_{ij} is the length or size of the common part of routes i and j.

The covariance of two random variates is

$$\rho(\epsilon_i, \epsilon_j) = \frac{\sigma(\epsilon_i, \epsilon_j)}{\sqrt{\sigma(\epsilon_i)}\sqrt{\sigma(\epsilon_j)}} \tag{3}$$

where $\rho(\epsilon_i, \epsilon_j)$ is the correlation coefficient and

$$\rho(\epsilon_i, \epsilon_j) = \frac{l_{ij}}{\sqrt{(L_i \cdot L_j)}} \tag{4}$$

where L_i and L_j are the length or size of routes i and j, respectively, which implies that other formulations of route correlation, such as Expressions 5 and 6, have to be considered as approximations, although they may have attractive properties.

$$\rho(\epsilon_i, \epsilon_j) = \frac{l_{ij}}{(L_i + L_j)/2} \tag{5}$$

$$\rho(\epsilon_i, \epsilon_j) = \frac{l_{ij}}{L_i} \tag{6}$$

In the case of more than two routes in the choice set, a straightforward extension of the correlation might be the ratio between total common lengths and total route lengths of the set. A measure for the correlation of a single route i with all the other routes in the choice set correspondingly is

$$\rho_i = \frac{\sum_{j \neq i} l_{ij}}{\frac{N-1}{N}\sum_j L_j} = \frac{\sum_{j \neq i}\sum_a l_a \cdot \delta_{ai} \cdot \delta_{aj}}{\frac{N-1}{N}\sum_j L_j} = \frac{\sum_a \delta_{ai} \cdot l_a \cdot (N_a - 1)}{\frac{N-1}{N}\sum_j L_j} \approx \frac{\sum_a \delta_{ai} \cdot l_a \cdot (N_a - 1)}{(N-1) L_i} \tag{7}$$

where

N = route set size,
N_a ($\leq N$) = number of routes using link a, and
δ_{ai} and δ_{aj} = link-path incidence between link a and routes i and j, respectively, equal to 1 when link a belongs to the respective route and 0 otherwise.

This expression is a measure for the path size (or independence) of route i relative to the other routes in the choice set. Assuming that all the routes in the choice set have nearly equal lengths, the following approximation for the multilateral correlation coefficient of route i is obtained:

$$\rho_i \approx \frac{1}{N-1}\sum_{j \neq i} \rho_{ij} \tag{8}$$

The correlation due to overlap is always positive.

Choice probabilities decrease with higher levels of correlation of a route relative to others. Intuitively, travelers perceive alternative routes less as distinct alternatives the more they overlap with other routes. The probability-reducing effect of the overlap may be demonstrated in a more strict way with a choice model. To that extent, the simple case of three routes, two of which (Routes A and B) overlap, is presented in Figure 1.

For this simple configuration an exact closed-form logit expression exists for the route choice probabilities (*5*):

$$p_{A,B} = \frac{\exp\left(\frac{\mu \cdot V_{A,B}}{1-\rho}\right)\left[\exp\left(\frac{\mu \cdot V_A}{1-\rho}\right) + \exp\left(\frac{\mu \cdot V_B}{1-\rho}\right)\right]^{-\rho}}{\left[\exp\left(\frac{\mu \cdot V_A}{1-\rho}\right) + \exp\left(\frac{\mu \cdot V_B}{1-\rho}\right)\right]^{1-\rho} + \exp(\mu \cdot V_C)} \tag{9}$$

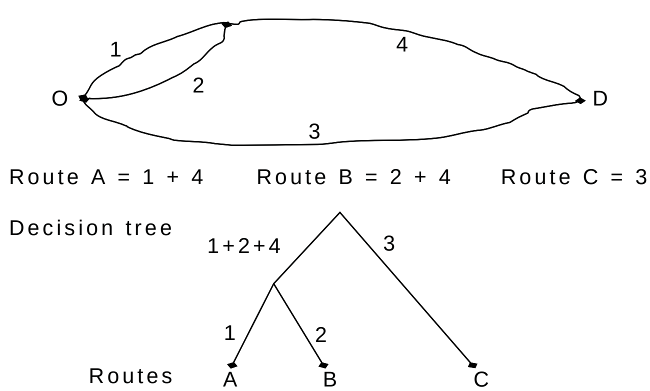

FIGURE 1 Example of overlapping problem.

$$p_C = \frac{\exp(\mu \cdot V_C)}{\left[\exp\left(\frac{\mu \cdot V_A}{1-\rho}\right) + \exp\left(\frac{\mu \cdot V_B}{1-\rho}\right)\right]^{1-\rho} + \exp(\mu \cdot V_C)} \tag{10}$$

where

$V_{A,B}$ and V_C = systematic route utilities of Routes A, B, and C, respectively,
μ = scale factor of elemental routes, and
ρ = similarity coefficient between overlapping Routes A and B ($0 \leq \rho \leq 1$).

If $\rho = 0$, there is no overlap; all three routes are independent, so the expression collapses into MNL. If $\rho = 1$, then Routes A and B completely overlap and are identical, so there are only two alternatives (see Equation 4).

The correlation ρ between two routes is equal for both routes as it is the ratio of the error term covariance, which equals the variance of the common part, and the error term variances of the two routes, assumed equal in the logit approach. This correlation can be plausibly approximated by the ratio of the lengths of the common part (l_a) and the routes (L):

$$\rho \approx \frac{l_a}{L} \tag{11}$$

Other measures for ρ are possible, but all are somehow a ratio of the common route part to the full routes.

From the probability expressions, it appears that increasing correlation leads to a smaller denominator in the probability function; thus, the independent alternative *C* becomes more probable. Consequently, the probabilities of the correlated alternatives diminish with the degree of their correlation.

This choice model can be used to assess the quality of approximation of the PSL vis-à-vis MNL and the exact logit probabilities, at least for this simple three-route case.

ALTERNATIVE DERIVATION OF PATH SIZE EXPRESSION

Frejinger and Bierlaire (*6*) offer the clearest motivation so far for the classic path size expression. Their derivation is based on the concept of aggregate alternative. The proposed reasoning follows the same logic, but, in contrast, results in a different expression that is corroborated below using a new derivation presented in the section on derivation of a new path size expression based on a nested logit (NL) model.

Size Factor of an Aggregate Alternative

Consider a network and a single origin–destination (O-D) pair. There is a set of elemental routes from origin to destination that may partly overlap. One is looking for the unique path size factor for a route utility function that accounts for the impact of the overlap in route choice.

C_n is defined as the set of paths between origin and destination considered by an individual, *n*. In this set, a nested structure is assumed based on common links, where each nest corresponds to an aggregate alternative grouping of the elemental alternatives (the routes). The aggregate subset corresponds to all elemental routes using a particular link *a*.

Figure 2 shows a typical pattern of aggregation of elemental routes (A to D) to subgroups characterized by their common links (1 to 5). Figure 2*a* shows all the individual links used by one or more routes (1 to 5). Figure 2*b* represents the elemental routes (A to D). Each route is connected to its links. If a link is used only by a single route, a route subgroup consists of one route. If a link is used by multiple routes, a route subgroup consists of the routes attached to that link.

C_{an} is defined as the subset of paths using link *a*, where $C_{an} \subset C_n$, and $a = 1, \ldots, N$, where *N* is the number of links in choice set C_n.

The utility U_{in} individual *n* associates with path *i* is

$$U_{in} = V_{in} + \epsilon_{in} \tag{12}$$

where V_{in} represents the deterministic part and ϵ_{in} is the random part of the utility.

For derivation of the path size one is interested in the choice of the elemental alternatives (route choice) by the traveler by considering the size of the aggregation of these alternatives by their common link *a*, equaling the number M_{an} of elemental paths using that link.

Size Factor for a Single Common Link in the Choice Set

Consider a single link *a* and the subgroup C_{an} of routes of size M_{an} using that link. For the time being, other possible overlaps among these paths and the other paths in C_n not using link *a* are neglected.

According to random utility choice theory, one can associate with such a subset of routes a size factor expressing an additional attractiveness of each alternative in this subgroup because of its size M_{an}. The larger a subgroup is, the higher is the probability that this subgroup and one of its members will be chosen by the traveler relative to other subsets of the same choice situation.

The subgroup utility U_{an} (sometimes confusingly called link utility in the literature) for individual *n* is defined by the maximum utility of routes *j* belonging to the subgroup (*7, 8*):

$$U_{an} = \max_{j \in C_{an}}\left(V_{jn} + \epsilon_{jn}\right) + \epsilon_{an} \qquad a = 1, \ldots, N \tag{13}$$

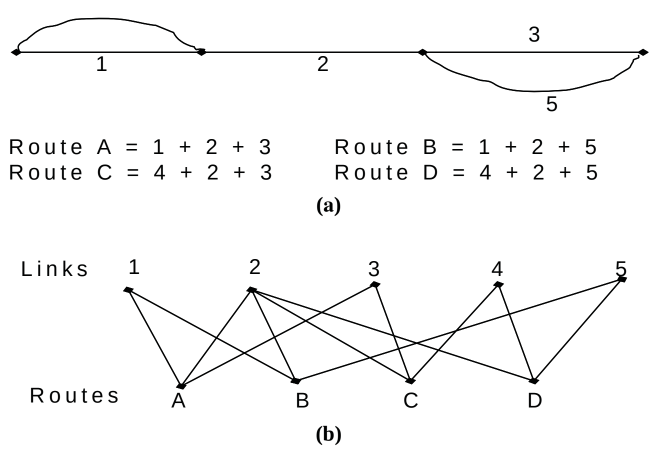

FIGURE 2 Example pattern of (*a*) overlapping routes and (*b*) their nested representation.

Utility U_{an} can also be expressed as the sum of its expectation V_{an} and its random term ϵ_{an}:

$$U_{an} = V_{an} + \epsilon_{an} \tag{14}$$

Assuming $E[\epsilon_{an}] = 0$, the deterministic component V_{an} is expressed as follows:

$$V_{an} = E\left[U_{an}\right] = E\left[\max_{j \in C_{an}}\left(V_{jn} + \epsilon_{jn}\right)\right] \tag{15}$$

The average deterministic utility of the paths using link a is defined as

$$\bar{V}_{an} = \frac{1}{M_{an}} \sum_{j \in C_{an}} V_{jn} \tag{16}$$

where M_a is the number of paths using link a (in the literature sometimes confusingly called size of link a), which can also be written as

$$M_a = \sum_{j \in C_n} \delta_{aj} \tag{17}$$

where δ_{aj} is the link-path incidence that equals 1 if link a is in path j and 0 otherwise.

According to random utility choice theory and several assumptions listed below, the utility an individual associates with the subgroup of routes using link a is equal to (*7*)

$$U_{an} = \bar{V}_{an} + \frac{1}{\mu_a} \ln M_{an} + \epsilon_{an} \tag{18}$$

where μ_a is the scale parameter of link *a*. The term $(1/\mu_a) \cdot \ln M_{an}$ represents the size factor being the extra utility because of the size of the subgroup. If $M_{an} = 1$, link a is used by only a single route; no overlap exists due to link a and this term vanishes. Scale parameter μ_a equals the variance parameter of the error term distribution ϵ_{an} of the subgroup of routes using link a and is always positive. The size factor $(1/\mu_a) \cdot \ln M_{an}$ therefore is also positive.

The following assumptions are used for this expression to be valid and arrive at a closed-form expression:

- The size of subset C_{an} is large (*8*);
- The path utilities of the elemental members of C_{an} have equal means V_{jn}; and
- The random terms ϵ_{jn} of all $j \in C_{an}$ are independent and identically distributed (iid), meaning that all routes j in C_{an} are independent, showing, for example, no other overlaps.

The first assumption is needed only if random distributions of alternatives are not Gumbel distributed, as only minimization of a large number of arbitrary distributions leads asymptotically to a Gumbel distribution.

Size Factor of an Elemental Alternative

Whereas the size factor is a positive correction for the size of an aggregate alternative to be added to each route of the subgroup, it is necessary to make a corresponding negative correction for finding the correct utility value of the elemental alternatives in the subgroup C_{an} if the choice model assumes mutual independence among the routes (such as MNL).

As $-\ln M_a = \ln(1/M_a)$, the size correction SC for elemental alternative j to be independent can therefore be defined as

$$\text{SC}_j = -\frac{1}{\mu_a} \ln M_{an} = -\frac{1}{\mu_a} \ln \sum_{j \in C_n} \delta_{aj} \tag{19}$$

This expression is always negative and thus makes a route less attractive.

To make this size expression insensitive to the link definition of the network and guarantee additivity across links in a route, the overlap between routes should not be expressed in links but in the length l_a of the common link(s). Furthermore, the common length needs to be related to the route length L_j to have a standardized expression for all routes in the choice set. This ratio at the same time is a measure for the level of correlation among the overlapping routes and leads to the following standardized size factor:

$$\text{SC}_j = -\frac{1}{\mu_a} \frac{l_a}{L_j} \ln M_{an} \tag{20}$$

If the common link a increases in length (at the cost of the total length), the size corrections increase as well. Plausibly, if $l_a = 0$, there is no size correction. If $l_a = L_j$ then $M_{an} = 1$ and there is no size correction either. This correction term looks very similar to one of C-logit's commonality factors (*7*).

In the classic derivation, however (*6*), the length standardization is done differently.

$$\text{PS}_j = \frac{1}{\mu_a} \ln\left(\frac{l_a}{L_j} \frac{1}{M_{an}}\right) \tag{21}$$

The problem with this expression is the extreme case of $l_a = 0$ for which this path size expression does not work.

Consider the case of only a single common link a in the choice set. The utility of each elemental route then is

$$U_{jn} = V_{jn} - \frac{1}{\mu_a} \frac{l_a}{L_j} \ln \sum_{j \in C_n} \delta_{aj} + \epsilon_{jn} \tag{22}$$

Correspondingly, the size factors for all other links of choice set C_n can be determined.

Size Factor for Multiple Common Links in the Choice Set

Introducing more than a single common link requires the following additional assumptions:

- All elemental routes of C_n that share links have the same systematic utility V_{in} and
- All error terms ϵ_{in} are iid and thus μ_a is equal to μ.

If a route has more than one link in common with other routes, the negative size corrections $(1/\mu_a) \cdot \ln M_{an}$ are simply weighted by their length and added, leading to the expression

$$U_{in} = V_{in} - \frac{1}{\mu} \sum_{a \in \Gamma_i} \frac{l_a}{L_i} \ln \sum_{i \in C_n} \delta_{aj} + \epsilon_{in} \tag{23}$$

The summation over links is natural given that the correlation is specified proportional to the common length (see the section on correlation among alternative routes).

The path size correction factor PSC_i of route i, which differentiates from the classic path size factor because of its derivation is defined as

$$\text{PSC}_i = -\frac{1}{\mu}\sum_{a\in\Gamma_i}\frac{l_a}{L_i}\ln\sum_{i\in C_n}\delta_{aj} \tag{24}$$

This expression shows that the lengths of the common links are weighted by the logarithm of the number of routes using these common links. This PSC specification has the following properties:

- The PSC value depends on the number of common links in the route, the lengths l_a of the common links, and the numbers M_a of distinct routes using each common link.
- A completely independent route without any common links has a maximum PSC value equal to 0.
- There is no lower bound on the PSC value, because there is no upper bound on M_a.
- The PSC specification is insensitive to the network specification.

The expression shows that the utility reduction increases with

- Increasing number of common links in a route,
- Increasing lengths of the common links, and
- Increasing number of other routes from the choice set that overlap with one or more links of the route.

The latter point also means that the path size (regardless of the specification) depends on the composition of the choice set, which has the unfortunate consequence that including unattractive routes in the choice set influences the choice probabilities of the attractive ones (*9*). This situation can be prevented only by using adequate choice sets in the modeling.

Note that the PSC expression significantly deviates from the classic one (*2, 6*). One reason for this difference follows from the unnecessary assumption in the classic path size derivation that the path size is a proportionality factor of the choice probability, as shown in the following expression:

$$p_i = \frac{\exp(V_i + \ln \text{PS}_i)}{\sum_{j\in C_n}\exp(V_j + \ln \text{PS}_j)} = \frac{\text{PS}_i \exp(V_i)}{\sum_{j\in C_n}\text{PS}_j \exp(V_j)} \tag{25}$$

DERIVATION OF A NEW PATH SIZE EXPRESSION BASED ON NESTED LOGIT MODEL

Another plausible way of deriving the PSC factor is to see it as a simplification and approximation for higher-level exact choice probability expressions in case of correlation among alternative routes. This derivation should yield the same expression as the one derived from the theory of aggregate alternatives. The demonstration accounts for the NL model as one of the more complex choice models. Similar derivations are possible with PCL and CNL, for example.

PSC Derivation from a Single-Nest Nested Logit Model

Consider the simple case of three routes, two of which (A and B) overlap (see Figure 1). For this simple configuration an exact closed-form logit expression exists, assuming iid error terms as presented in Equations 9 and 10 (*5*).

Starting with the exact probability expression given in Formulas 9 and 10, a path size expression is derived by applying the assumptions listed in the section on alternative derivation of path size expression. By assuming equal systematic utilities $V_{A,B}$ for the correlated alternatives A and B, the probability expressions reduce to

$$p_{A,B} = \frac{\exp(\mu\cdot V_{A,B} - \rho\ln M_a)}{\sum_{A,B}\exp(\mu\cdot V_{A,B} - \rho\ln M_a) + \exp(\mu\cdot V_C)} \tag{26}$$

$$p_C = \frac{\exp(\mu\cdot V_C)}{\sum_{A,B}\exp(\mu\cdot V_{A,B} - \rho\ln M_a) + \exp(\mu\cdot V_C)} \tag{27}$$

or more generally

$$p_i = \frac{\exp\left(\mu\cdot V_i - \rho\ln\sum_i\delta_{ai}\right)}{\sum_j\exp\left(\mu\cdot V_j - \rho\ln\sum_j\delta_{aj}\right)} \tag{28}$$

which corresponds to the PSC expression from the section on alternative derivation of path size expression.

In this simple case, the impact of correlation on the ratio of probabilities of correlated (Route A) and noncorrelated (Route C) alternatives can be described by an extra term expressing the reduction of attractiveness of the correlated alternative. In this term, the PSC expression can be recognized when one substitutes the correlation factor $\rho \approx l_a/L$.

Consider the calculated route choice probabilities for this small network by comparing the exact solution, the classic path size factor, and the new PSC factor. The relative size of the overlap is assumed to range from zero (no overlap and thus three independent routes) to one (complete overlap and thus only two independent routes). The PSC expression shows that, as long as the disutilities of the overlapping routes are equal, the PSC results are identical to the exact ones. In that case, the PSL model logically shows a bias in that the probabilities of the correlated alternatives are systematically overestimated at the cost of the underestimated independent alternative.

Table 1 presents the probabilities of the noncorrelated Route C with disutility equal to 2.2, compared with disutility equal to 1.8 and 2 for correlated Routes A and B, respectively. Parameter β and scale factor μ are both set equal to 1 for all four models.

TABLE 1 Route Choice Probabilities of Uncorrelated Route Depending on Relative Size of Overlap

Overlap	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Exact	.269	.283	.297	.312	.326	.342	.357	.372	.386	.398
MNL	.269	.269	.269	.269	.269	.269	.269	.269	.269	.269
PSL	.269	.280	.292	.304	.318	.333	.349	.368	.388	.410
PSC-L	.269	.283	.296	.310	.324	.339	.354	.369	.385	.401

In the case of unequal disutility, both path size models result in biased estimates for the correlated alternatives (the shorter one too low and the longer one too high) with increasing bias at higher levels of overlap. Both path size models give fairly good estimates for the independent alternative, although apparently somewhat better by the PSC-logit model.

PSC Derivation from a Multinest Nested Logit Model with a Single-Scale Parameter

Consider the more general case of a NL model with a set of K multiple common links (indexed a) among the routes showing a hierarchical tree structure (Figure 3).

Each route belongs to a single subgroup C_a of routes belonging to the same nest defined by common link a (Links 3 and 4). The route subsets are nonoverlapping.

The case with a single-scale parameter μ is derived, and the scale factor for the nests is assumed equal and set to 1. The choice probability is then defined by (*8*)

$$p_i = \frac{\exp(V_i)\left(\sum_{i\in C_a}\exp(V_i)\right)^{\mu-1}}{\sum_{i\in C_a}\exp(V_i)\sum_{a\in K}\left(\sum_{i\in C_a}\exp(V_i)\right)^{\mu}} \qquad (29)$$

Assuming again equal V_i within the nests, the choice probability is

$$p_i = \frac{\exp\mu\left(V_i \frac{1-\mu}{\mu}\ln M_a\right)}{\sum_{a\in K}\exp\mu\left(V_a+\ln M_a\right)} \qquad (30)$$

The numerator refers to the elemental route i of subset a, and the denominator sums over all nests.

PSC Derivation from a Multinest Nested Logit Model with Two-Scale Parameters

Consider the same general case of a NL model (Figure 3), but assume different scale factors for the nests. The choice probabilities of the elemental alternatives (routes) are defined by (*7, 8*)

$$p_i = \frac{\exp(\mu_a V_i)}{\sum_{j\in C_a}\exp(\mu_a V_j)}\frac{\exp(\lambda_a Y_a)}{\sum_{b\in K}\exp(\lambda_b Y_b)} \qquad (31)$$

where λ_a and λ_b are the scale factors for nests a and b, respectively, and Y_a and Y_b are their logsums. In this NL model, the error term variances μ_a of the subgroup C_a of routes in nest a (and therefore also correlation coefficients ρ_a) may be specific for each common link a. The larger the correlation among the routes, the larger is the value of μ_a being the measure of similarity within the subset.

This NL model can be reduced to a precursor of the PSC-logit model, resulting in the following expression where the numerator refers to the elemental route i of subset a, and the denominator sums over all subsets of routes.

$$p_i = \frac{\exp\left(\mu\cdot V_i-(1-\lambda_a)\ln M_a\right)}{\sum_{b\in K}\exp\left(\mu\cdot\bar{V}_b+\lambda_a\ln M_a\right)} \qquad (32)$$

Parameter $(1-\lambda)=1-\sqrt{1-\rho}$ is a measure for the level of correlation among the routes in a subset if overlapping, with $0\le(1-\lambda)\le 1$. The larger the correlation ρ, the larger is the value of $(1-\lambda)$. Consequently, the reduction in utility of an alternative due to overlap with others because of a single common link is approximately proportional to the level of overlap weighted by the number of overlapping routes.

DISCUSSION OF THE PATH SIZE CORRECTION FACTOR

The path size specifications given in the literature and derived here are approximations based on the following strict assumptions required for their derivation:

- Equality of deterministic utility V_{in} across all overlapping routes,
- Specification for the correlation between the routes, and
- No other correlations or interdependencies other than the overlap.

In addition, the way of weighting the commonality among routes is arbitrary to a certain extent, although physical length appears more natural than the other possibilities (e.g., travel time).

The attempts to derive the path size for the MNL model by simplifying more complex models that somehow address the correlation among alternatives (NL, PCL, CNL, etc.) derive different

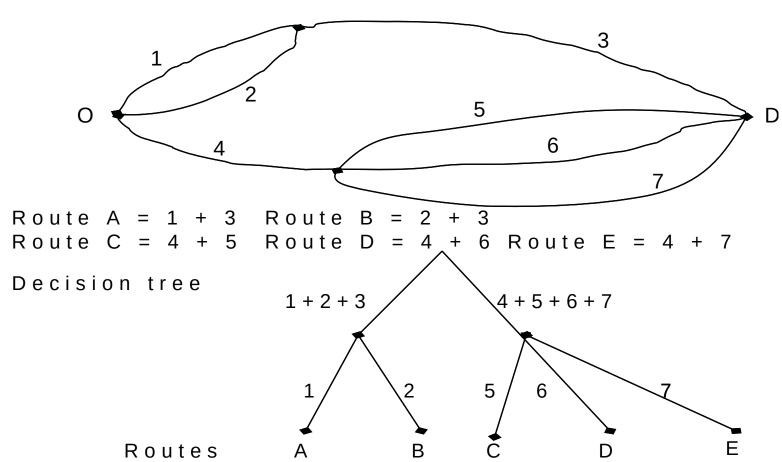

FIGURE 3 Route pattern with multiple disjoint nests of overlapping routes.

specifications of the path size, most likely because these models assume different correlation structures.

Given these different possible path size specifications, choosing the best of them is partly a matter of theoretical analysis (which one can address correlations best) and partly a matter of empirical analysis (which specification gives the best estimation results).

Because the PSC factor derived above holds only for very strict assumptions about the choice set and depends on arbitrarily chosen weighting, the given PSC specification is to be considered a heuristic approximation to the impact of route overlap on route choice. The same holds for classic path size factor. In addition, apart from the purely statistical size considerations following from random utility choice theory, overlap among routes may have behavioral (perception) impacts. For all these reasons, the impact of the PSC factor on choice behavior should be described by a parameter β_{PS} to be estimated from observations (*3*).

The completed route utility function thus looks as follows:

$$U_{in} = V_{in} - \beta_{PS} \sum_{a \in \Gamma_i} \frac{l_a}{L_i} \ln \sum_{i \in C_n} \delta_{aj} + \epsilon_{in} \tag{33}$$

The scale factor μ in the path size factor disappears by the formulation of the PSC-logit choice model, because

$$\begin{aligned} p_{in} &= \frac{\exp\left[\mu\left(V_{in} - \beta_{PS}\frac{1}{\mu}\sum_{a\in\Gamma_i}\frac{l_a}{L_i}\ln M_{an}\right)\right]}{\sum_{jn\in C_n}\exp\left[\mu\left(V_{jn} - \beta_{PS}\frac{1}{\mu}\sum_{a\in\Gamma_j}\frac{l_a}{L_j}\ln M_{an}\right)\right]} \\ &= \frac{\exp\left[\mu V_{in} - \beta_{PS}\frac{1}{L_i}\sum_{a\in\Gamma_i} l_a \ln M_{an}\right]}{\sum_{jn\in C_n}\exp\left[\mu V_{jn} - \beta_{PS}\frac{1}{L_j}\sum_{a\in\Gamma_j} l_a \ln M_{an}\right]} \end{aligned} \tag{34}$$

with expectedly, but not necessarily, a positive parameter β_{PS}.

APPLICATIONS

To test whether the alternative PSC expression makes a difference in choice modeling, route choice models were estimated for two urban networks: the U.S. city of Boston and the Italian city of Turin. In addition, choice predictions were calculated for a synthetic network with a known choice set.

Comparison of Model Estimates

Model estimation focuses on the networks of Boston and Turin. The Boston network consists of 800 zones, 12,190 nodes, and 19,148 links. A total of 188 routes connecting 91 O-D pairs were collected among faculty and staff members during the 1997 Transportation Survey for MIT employees (*10*). The Turin network consists of 23 districts, 92 zones, 417 nodes, and 1,427 links. A total of 236 routes relating 182 O-D pairs were collected among faculty and staff members of Turin Polytechnic who volunteered to participate in a web-based questionnaire (*11*).

Choice sets are generated with a branch-and-bound algorithm that constructs a connection tree between origin and destination of each observed route, by processing sequences of links according to a branching rule that accounts for behavioral constraints (*12*). For both networks, these behavioral constraints cause rejection from the choice set paths that take the driver 10% of the distance farther from the destination and closer to the origin, contain detours larger than 120% with respect to alternative paths, involve difficulty in being distinguished because of their high similarity (80% or more) with other paths. For the Turin network, the behavioral constraints lead to discard paths that constitute unrealistic options because of being 150% longer than the optimal path in terms of travel time or consider more than four maneuvers causing delays to traffic circulation. In Boston, the observed routes are longer; consequently, comparable travel time excess with respect to the shortest path and analogous maneuver availability are obtainable through the definition of a lower temporal constraint (133%) and a higher movement constraint (seven maneuvers).

Path sets constructed with the branch-and-bound algorithm are compared with the actual chosen routes, and observations that are not reproduced with an 80% overlap threshold are excluded from consideration for both networks to have choice sets consistent with the actual behavior. Accordingly, the Turin choice set contains 228 observations and the Boston data set includes 181 observations.

Utility functions are defined through the definition of the following route attributes: DIST is the total route length in kilometers, TT is the total route travel time in minutes, TMRPC is the percent of travel time on major roads, FSPD is the dummy variable for the path with the maximum average speed, DELPC is the percent delay with respect to the free flow time, and LOGPS is the path size term. FSPD is present in the Boston specification, as for longer routes the average speed is more significant, whereas DELPC is considered in the Turin network where congestion plays a relevant role in an urban environment.

Model estimates for MNL, PS-logit, and PSC-logit are illustrated in Table 2. MNL estimates are presented as benchmark terms to evaluate the performance of the enhanced models, and all the estimation processes are performed with Biogeme (*13*). For ease of comparison with the MNL model, the parameters were scaled so that the travel time coefficient TT is equal in all models. The scale factors for the Turin network are 1.0, 0.80, and 0.66 for all the models.

Parameter estimates across different model structures do not differ within the same data source. Parameter values illustrate that drivers tend to minimize distance as well as travel time, tend to travel on major roads, attempt to avoid highly congested roads in the Turin network, and seek to increase the average speed in the Boston network.

Distance parameter values suggest that drivers most likely perceive distance differences as more influential on their choices in a smaller network than in a larger network, and the value from the joint estimation appears similar to the Boston model.

Path size terms exhibit the expected positive sign, which confirms the theoretical assumption that the utility of similar alternatives is reduced with respect to the utility of unique alternatives.

It appears that adding a path size term affects the Turin parameters much more (especially travel time) than in the Boston case. Otherwise, the differences in parameter values, goodness of fit, and predictive ability (*8*, pp. 91–92) between the two path size specifications are fairly small in both cases.

Comparison of Predicted Choice Probabilities

Consider the following simple network (Figure 4) with 16 unidirectional links, nine nodes, and a single O-D pair. All links have the same systematic disutility equaling 1. In this network, there are 12 routes with differing disutility equaling four units (Routes 1 to 6), six units (Routes 7 to 10), and eight units (Routes 11 and 12).

The degree of overlap of a route with all others is expressed by the total overlap length divided by the total route lengths of all routes in the choice set (which is a correlation coefficient).

An exact solution, derived from probit simulation, is compared with predictions from the MNL model, PSL model, and PSC-logit

TABLE 2 Estimation Results for Turin and Boston Networks

	MNL			PS-LOGIT			PSC-LOGIT		
Model Parameter	Value	Scaled	*t*-Test	Value	Scaled	*t*-Test	Value	Scaled	*t*-Test
Turin Dataset									
DIST	–1.268	–1.268	–6.01	–1.155	–0.926	–5.72	–0.923	–0.610	–3.26
TT	–0.295	–0.295	–4.86	–0.368	–0.295	–6.03	–0.447	–0.295	–5.29
TMRPC	0.204	0.204	9.04	0.175	0.140	7.77	0.146	0.096	4.58
DELPC	–0.561	–0.561	–2.02	–0.703	–0.564	–2.52	–0.761	–0.503	–2.01
LOGPS				0.928	0.744	4.38	1.045	0.691	2.71
Parameters estimated		4			5			5	
Observations		228			228			228	
Log likelihood at zero		–616.27			–616.27			–616.27	
Log likelihood at estimates		–504.06			–495.03			–496.93	
Adjusted rho-bar squared		0.176			0.189			0.186	
Prediction (“% right”)		28.51			30.26			31.14	
Boston Dataset									
DIST	–0.090	–0.090	–2.52	–0.094	–0.095	–2.51	–0.093	–0.092	–2.59
TT	–0.280	–0.280	–6.37	–0.279	–0.280	–6.28	–0.282	–0.280	–6.38
TMRPC	0.107	0.107	4.61	0.107	0.107	4.60	0.107	0.107	4.62
FSPD	1.411	1.411	6.61	1.394	1.401	6.22	1.389	1.389	6.34
LOGPS				0.642	0.646	2.40	0.802	0.796	2.10
Parameters estimated		4			5			5	
Observations		228			181			181	
Log likelihood at zero		–587.22			–587.22			–587.22	
Log likelihood at estimates		–498.61			–493.58			–493.52	
Adjusted rho-bar squared		0.144			0.151			0.151	
Prediction (“% right”)		24.86			25.97			27.07	

model. In all models, the disutility parameter β and the scale factor μ are set equal to 1.

Consider the full choice set of 12 routes. In the full choice set, the degree of overlap is largest for Routes 11 and 12 (least independence) and smallest for Routes 3 to 6 (highest independence), which implies that the latter routes have the highest choice probabilities in the PSL and PSC-logit models, and both models show poor performance in replicating the exact values. The differences between the PSL and PSC-logit model are marginal, as the problem is the high overlap between unattractive routes and attractive ones, which leads to biased path size values.

Consider the subset of the most attractive routes only—namely, Routes 1 to 6. In this much smaller and more homogeneous choice set, the degree of overlap strongly differs from the full set and has

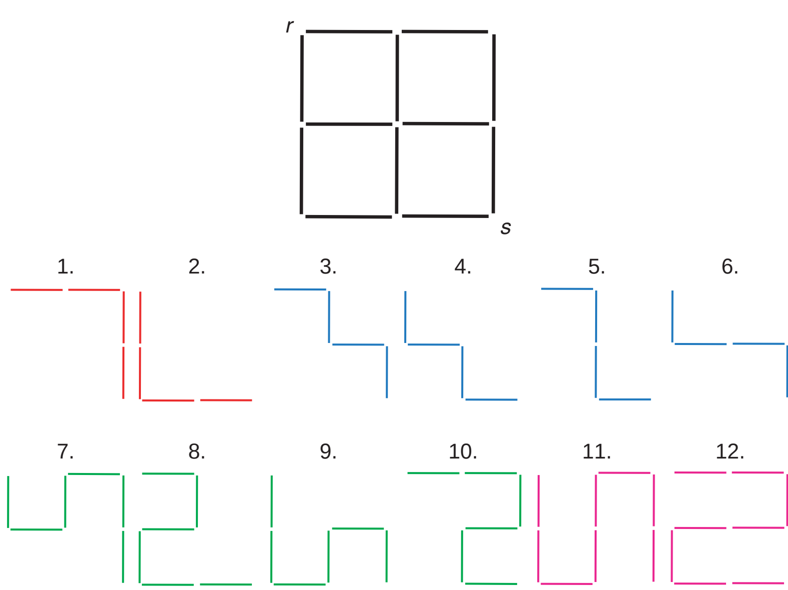

FIGURE 4 Network with all available routes from *r* to *s*.

TABLE 3 Choice Probability Predictions of Full Route Choice Set

Route	Length	Overlap	Exact	MNL	PSL	PSC-Logit
Full Choice Set						
1	4	.344	.196	.152	.121	.126
2	4	.344	.196	.152	.121	.126
3	4	.281	.146	.152	.165	.163
4	4	.281	.146	.152	.165	.163
5	4	.281	.154	.152	.165	.163
6	4	.281	.154	.152	.165	.163
7	6	.422	.003	.021	.023	.022
8	6	.422	.003	.021	.023	.022
9	6	.422	.003	.021	.023	.022
10	6	.422	.003	.021	.023	.022
11	8	.563	0	.003	.003	.003
12	8	.563	0	.003	.003	.003
Reduced Choice Set						
1	4	.25	.196	.167	.222	.207
2	4	.25	.196	.167	.222	.207
3	4	.42	.146	.167	.139	.146
4	4	.42	.146	.167	.139	.146
5	4	.42	.153	.167	.139	.146
6	4	.42	.153	.167	.139	.146

the reverse order, which in turn leads to corresponding shifts in the calculated path size model choice probabilities—which look now much closer to the exact ones. Given the assumed equality of parameter values in both path size expressions, the PSC values appear to agree better with the exact ones than was achieved with the classic PSL model.

Table 3 summarizes the results for both full and reduced choice sets. It is derived from this exercise that PSL and PSL-logit models should be applied only to choice sets including all attractive routes and neglecting the unattractive ones (*9*).

SUMMARY AND CONCLUSIONS

The path size of a route is a concept that tries to capture the influence of overlap among alternative routes on choice probabilities. The path size measure is useful in partly removing some drawbacks of the MNL model in the case of correlations among routes in a choice set. Using the relatively simple PSL model is a practical alternative to the more complex choice models (such as PCL, CNL, logit-kernel, and probit) that can more correctly handle correlations due to overlap, although with additional effort.

The main contribution of this paper is the detailed and systematic derivation of a formulation of the path size measure and the explicit definition of its assumptions and properties. The new expression results from the notion of aggregate alternative and also from simplifying higher-order logit models that explicitly capture correlations. Also, the new expression uses the same topologic properties and variables as the classic one but in a different computational form. Last, the new measure offers a more natural interpretation of the role of correlation due to spatial overlap. In some simple hierarchical networks, the new path size measure gives results identical to the exact ones. The new derivation explicitly shows which approximating assumptions are needed to derive the path size measure.

From the estimation findings on two real-world networks the PSC-logit model performs as well as the PSL. From calculation of predicted choice probabilities in synthetic networks, the PSC factor appears to outperform the classic path size factor. However, calculation of the predicted choice probabilities in the two networks (Boston and Turin) shows only a marginal improvement. These applications also show that the path size models appear to be highly sensitive to the composition of the route choice set and can successfully be applied only to choice sets freed from unrealistic routes. Further research needs to be done to include better consideration of route set selection probabilities, which in turn will affect the choice probabilities.

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