International Journal of Structural Stability and Dynamics
Vol. 3, No. 1 (2003) 1–16


# STABILITY CONJECTURE IN THE THEORY OF TENSEGRITY STRUCTURES

K. Y. VOLOKH
*Department of Civil and Environmental Engineering,*
*Technion — Israel Institute of Technology, Haifa 32000, Israel*
*cvolokh@tx.technion.ac.il*

Received 26 August 2002
Accepted 9 January 2003

The general problem of the stability of tensegrity structures comprising struts and cables is formulated. It is conjectured that any tensegrity system with totally tensioned cables is stable independently of its topology, geometry and specific magnitudes of member forces.

*Keywords*: Stability; tensegrity; cables.

## 1. Introduction

Tensegrity is an abbreviation of "tension" + "integrity". It designates a class of truss structures, which consists of pin-jointed compressed struts and tensioned cables. There is only one compressed strut at every node of the assembly while the number of tensioned cables is arbitrary.[9,24,32] The first tensegrity structure was invented by Kenneth Snelson in 1948.[26] Buckminster Fuller named the Snelson structure "tensegrity" and proposed a new class of tensegrity domes.[19] The most impressive application of tensegrity systems in construction is Geigers tensegrity dome proposed in designing the Olympic arena in Seoul, Korea.[10] Domes of this type were used also in building the Redbird Arena at Illinois State University, and the Sun Coast Dome in St. Petersburg, Florida in the United States. Instances of natural occurrence of tensegrity structures vary from deployable grids[8,11,20] to cytoskeletons of living cells.[7,12,37,38]

The increasing interest in mechanics of tensegrity structures started from the celebrated paper by Calladine[2] where the singular character of tensegrities was emphasized. It was realized since then that tensegrity structures are both statically and *kinematically* indeterminate.[28] The latter means that infinitesimal displacement modes exist that do not produce strains.[35] Kinematically indeterminate structures are also called *underconstrained structures* or infinitesimal (instantaneous) mechanisms or singular structures.[16,17] The consideration of kinematically indeterminate structures requires an extension of the traditional classification of structures.

It is worth noting that the kinematically indeterminate structures are often misunderstood in the engineering literature: they are considered as mechanisms unable bearing loads. This is inaccurate, of course. The exciting problem of the classification of kinematically indeterminate structures attracted attention of many researchers: Tarnai,[28–30] Koiter,[15] Calladine and Pellegrino,[3,4] Salerno,[25] Vassart *et al.*[31] and Kuznetsov.[16] The kinematical indeterminacy generally leads to the necessity to account for geometrical nonlinearity in static and dynamic analyses of underconstarined structures including tensegreties. An extensive work has been performed in this direction by Kebiche *et al.*[13] Murakami,[21,22] Oppenheim and Williams,[23] Wang,[40] Yuan and Dong,[41] Volokh[33,34] and Volokh *et al.*[39]

The intriguing feature of tensegrity structures is their stability at the initial self-stress state (prestressability) in the known computational and practical examples.[18,27] The latter raises general theoretical question: *are all tensegrity assemblies with tensioned cables and compressed struts stable independently of their topology, geometry and specific magnitudes of member forces?* The positive answer to this question is conjectured in this note. For this purpose, the general problem of stability of tensegrity structures is formulated. It is assumed that all structural members are straight and undergo large rigid body motions while the axial strains are small and the displacement distribution along the members is linear. Hooke's law is adapted as the constitutive equation. The local buckling of struts and compression of cables are excluded. Results on the stability of pre-tensioned cable nets are used to motivate the conjecture. The relevant mathematical issues are emphasized. It is believed that the possible proof of the conjecture could explain the observed phenomena of the prestressability of tensegrity structures.

## 2. Formulation of the Stability Problem

The structure is stable if the equilibrium state corresponds to a minimum of the potential energy, which is attained for kinematically admissible displacement ($\mathbf{u}$) variations:[1,14]

$$\{\psi(\mathbf{u}+\delta\mathbf{u}) - \psi(\mathbf{u})\}/\|\delta\mathbf{u}\|^2 \geq c > 0\,. \tag{1}$$

The potential energy under "dead" tractions can be generally written in the form:

$$2\,\psi = \int_{V_0} (\mathbf{S} : \mathbf{E}/2 + \mathbf{S}_0 : \mathbf{E})\,dV_0 - \int_{A_0} (\mathbf{t} + \mathbf{t}_0) \cdot \mathbf{u} dA_0 \tag{2}$$

$$\mathbf{S} = \mathbf{C} : \mathbf{E} \tag{3}$$

$$\mathbf{E} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2 + \nabla\mathbf{u} \cdot \nabla\mathbf{u}^T/2\,. \tag{4}$$

The Hooke's law with the fourth-order tensor of elastic moduli $\mathbf{C}$ is used because the Green–Lagrange strain tensor is assumed to be small $\|\mathbf{E}\| < \|\mathbf{1}\|$, that is axial and angle changes of the material wires are small while large rotations and translations are permitted. Tensors $\mathbf{S}$ and $\mathbf{S}_0$ designate the second Piola–Kirchhoff stress increments and initial stresses accordingly ant vectors $\mathbf{t}$ and $\mathbf{t}_0$ designate traction increments and initial tractions, which are in equilibrium with initial stresses. The initial tractions are zero at the state of self-stress. Displacements in Eq. (1) are obtained from the equilibrium condition, accounting for (3) and (4):

$$\delta\psi = \int_{V_0} \{(\mathbf{S} + \mathbf{S}_0) : \delta\mathbf{E}\}\, dV_0 - \int_{A_0} (\mathbf{t} + \mathbf{t}_0) \cdot \delta\mathbf{u} dA_0 = 0\,. \tag{5}$$

Equations (1)–(5) cover the stability problem for a wide range of elastic structures. The general relations should be further specified for a particular class of structures using additional geometrical and physical assumptions of “engineering theories”.

In the case of tensegrity systems or general pin-jointed structures every member of an assembly is considered as a straight strut undergoing small axial strain and large rigid body motion. The latter means that the strain tensor of the $n$th member of the assembly contains only one entry:

$$E_{\xi\xi} = \frac{\partial u_\xi}{\partial \xi} + \frac{1}{2}\left\{\left(\frac{\partial u_\xi}{\partial \xi}\right)^2 + \left(\frac{\partial u_\eta}{\partial \xi}\right)^2 + \left(\frac{\partial u_\vartheta}{\partial \xi}\right)^2\right\}. \tag{6}$$

The axis $\xi$ of the local Cartesian coordinate frame $\xi\eta\vartheta$ is chosen along the $n$th member. Assuming the linear distribution of displacements along the member, the derivatives take the form:

$$\frac{\partial u_\xi}{\partial \xi} = \frac{\Delta u_\xi}{L}\,; \quad \frac{\partial u_\eta}{\partial \xi} = \frac{\Delta u_\eta}{L}\,; \quad \frac{\partial u_\vartheta}{\partial \xi} = \frac{\Delta u_\vartheta}{L} \tag{7}$$

$L$ designates the initial length of the member; and displacement increments are equal to the difference between the nodal (edge) displacements of the element. Accounting for (7), Eq. (6) takes the form:

$$E_{\xi\xi} = \frac{\Delta u_\xi}{L} + \frac{1}{2L^2}\left\{(\Delta u_\xi)^2 + (\Delta u_\eta)^2 + (\Delta u_\vartheta)^2\right\}. \tag{8}$$

This equation can be rewritten in a global Cartesian frame $xyz$:

$$\begin{aligned} E_{\xi\xi} &= \frac{1}{L}\left(\Delta u \cos(\xi x) + \Delta v \cos(\xi y) + \Delta w \cos(\xi x)\right) \\ &\quad + \frac{1}{2L^2}\left\{(\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2\right\}. \end{aligned} \tag{9}$$

The term in the braces is invariant. The displacement increment and direction cosines can be expressed in terms of coordinates and displacements of the $i$th and $j$th nodes of the member:

$$\Delta u = u_j - u_i\,; \quad \Delta v = v_j - v_i\,; \quad \Delta w = w_j - w_i \tag{10}$$

$$\begin{aligned} \cos(\xi x) &= \frac{\Delta x}{L} = \frac{x_j - x_i}{L} \\ \cos(\xi y) &= \frac{\Delta y}{L} = \frac{y_j - y_i}{L} \\ \cos(\xi z) &= \frac{\Delta z}{L} = \frac{z_j - z_i}{L}\,. \end{aligned} \tag{11}$$

Finally, the member strain takes the canonical form:

$$E_{\xi\xi} = \frac{1}{L^2}\left\{\Delta u\Delta x + \Delta v\Delta y + \Delta w\Delta z + (\Delta u)^2/2 + (\Delta v)^2/2 + (\Delta w)^2/2\right\}. \tag{12}$$

The first and the second varitions of this strain take the form:

$$\delta E_{\xi\xi} = \frac{1}{L^2}\left\{(\Delta x + \Delta u)\delta[\Delta u] + (\Delta y + \Delta v)\delta[\Delta v] + (\Delta z + \Delta w)\delta[\Delta w]\right\} \tag{13}$$

$$\delta^2 E_{\xi\xi} = \frac{1}{L^2}\left\{(\delta[\Delta u])^2 + (\delta[\Delta v])^2 + (\delta[\Delta w])^2\right\} \tag{14}$$

$$\delta[\Delta u] = \delta u_j - \delta u_i;\ \delta[\Delta v] = \delta v_j - \delta v_i;\ \delta[\Delta w] = \delta w_j - \delta w_i\,. \tag{15}$$

Substituting (13) in (5) and designating the conjugate stress entries $S$ and $S_0$ it is possible to obtain:

$$\begin{aligned}
\delta\psi_n &= \int_{V_n} (S + S_0)\delta E_{\xi\xi} dV_n \\
&= \{-(S+S_0)A_0(\Delta x + \Delta u)/L\}\delta u_i + \{(S+S_0)A_0(\Delta x + \Delta u)/L\}\delta u_j \\
&\quad + \{-(S+S_0)A_0(\Delta y + \Delta v)/L\}\delta v_i + \{(S+S_0)A_0(\Delta y + \Delta v)/L\}\delta v_j \\
&\quad + \{-(S+S_0)A_0(\Delta z + \Delta w)/L\}\delta w_i + \{(S+S_0)A_0(\Delta z + \Delta w)/L\}\delta w_j \\
&= 0\,.
\end{aligned} \tag{16}$$

The coefficients in braces are contributions of axial member forces of the $n$th element to the equilibrium of the $i$th and $j$th nodes. It is assumed that the stresses are distributed uniformly within the cross-section area and the tractions are concentrated at the nodes and not included in (16). Equation (16) allows for simple interpretation of components of the second Piola–Kirchhoff tensor. Let the coefficient of the first term be written in the form:

$$-\left[(S + S_0)A_0\frac{l}{L}\right]\left\{\frac{\Delta x + \Delta u}{l}\right\}$$

where $l$ designates the member length after the deformation. The expression in *braces* is the direction cosine for the deformed configuration. Thus the expression in *brackets* is the axial force. Since the axial stretch is small ($l/L \sim 1$) the increments of the second Piola–Kirchhoff stress can be interpreted approximately as the axial stresses at the current configuration. Assuming displacements and stress increments to be zero, the initial stress $S_0$ is exactly the axial stress at the initial configuration. The second variation of the potential energy for the $n$th member takes the form:

$$\begin{aligned}
\delta^2\psi_n &= \int_{V_n} \{\delta S\delta E_{\xi\xi} + (S + S_0)\delta^2 E_{\xi\xi}\}\, dV_n \\
&= \int_{V_n} \{E(\delta E_{\xi\xi})^2 + (S + S_0)\delta^2 E_{\xi\xi}\}\, dV_n \\
&= EA_0L(\delta E_{\xi\xi})^2 + (S + S_0)A_0L\delta^2 E_{\xi\xi}
\end{aligned} \tag{17}$$

$E$ designates Young's modulus. To find the quadratic form of the second variation of the potential energy it remains to substitute (13) and (14) in (16). The displacements and stress increments should be dropped from the final expression since the stability of the initial state is considered. The latter does not affect generality, of course, because any current state can be referred to as the "initial" one.

The stability criterion (1) takes the following form in the case of pin-jointed assemblies:

$$\delta^2\psi = \delta^2 \sum_{n=1}^{N} \psi_n = \sum_{n=1}^{N} \delta^2\psi_n = \delta\mathbf{u}^T \mathbf{K} \delta\mathbf{u} > 0\,. \tag{18}$$

The dimension $M$ of the vector of virtual displacement equals the number of degrees of freedom after excluding supporting points. The tangent stiffness matrix can be presented in the form:

$$\mathbf{K} = \mathbf{B}^T\mathbf{C}\mathbf{B} + \mathbf{D}\,. \tag{19}$$

The first and the second terms on the right hand side of (19) are the result of assembling terms $EA_0L(\delta E_{\xi\xi})^2$ and $S_0A_0L\delta^2E_{\xi\xi}$ over all members accordingly. An $N$ by $M$ matrix $\mathbf{B}$ is a standard matrix of direction cosines: $B_{nm} = (X_m - X_k)/L_n$. Letters $X$ designate nodal coordinates, which may be $x$ or $y$ or $z$; $n$ is the member number; $m$ and $k$ are proper indexes of the member coordinates. Diagonal matrix $\mathbf{C}$ is an $N$ by $N$ uncoupled stiffness matrix: $C_{nn} = E_nA_{0n}/L_n$. An $M$ by $M$ symmetric matrix $\mathbf{D}$ is the geometric stiffness matrix, whose entries take the form: $D_{mm} = \sum_{n=1}^{s} P_n/L_n$; $D_{mt} = -P_n/L_n$; $P_n = (S_0A_0)_n$. The sum includes all $(s)$ members attached to the node with the $m$th degree of freedom. Index $t$ is the properly chosen degree of freedom at the second edge of the $n$th member. It is important to emphasize that matrix $\mathbf{K}$ has no rows and columns corresponding to the constraints imposed by supports.

To summarize: *the considered initial state is stable if the tangent stiffness matrix* $\mathbf{K}$ *is positive definite*. It should not be missed that the theory above considers structural members to be straight. If all cables are tensioned and all struts do not buckle locally the straightness assumption seems to be valid. This is not always the case, however. Cable members do not resist compression and struts may buckle.[34]

It is important to emphasize that the stability formulation given above is typical of the Structural Mechanics approach. Another view of tensegrity and stability is promulgated by Connelly and his co-workers in the mathematical literature (see Connelly and Back,[5] Connelly and Whiteley,[6] and references therein). These authors do not restrict the definition of tensegrities to *one* compression strut at a joint and they use methods of graph, group, and representation theories. An interesting result brought by these authors will be considered in the next section.

## 3. Stability Conjecture

To motivate the following stability conjecture we first consider totally tensioned cable nets. The stability of tensioned cable nets was proved by Volokh and

Vilnay.[36] Three main steps of this proof include: (1) positive semi-definiteness of $\mathbf{B}^T\mathbf{CB}$; (2) positive semi-definiteness of $\mathbf{D}$; (3) non-singularity of $\mathbf{D}$. Positive semi-definiteness of $\mathbf{B}^T\mathbf{CB}$ is a direct consequence of the diagonal structure of $\mathbf{C}$ with positive entries. The latter allows for decomposition: $\mathbf{C} = \sqrt{\mathbf{C}}^T\sqrt{\mathbf{C}}$. Now the corresponding quadratic form can be written as follows: $\delta\mathbf{u}^T\mathbf{B}^T\sqrt{\mathbf{C}}^T\sqrt{\mathbf{C}}\mathbf{B}\delta\mathbf{u} = \|\sqrt{\mathbf{C}}\mathbf{B}\delta\mathbf{u}\|_2^2 \geq 0$. It is important to emphasize that in the case of kinematically indeterminate structures matrix $\mathbf{B}$ possesses a nontrivial nullspace formed by vectors usually called infinitesimal mechanisms modes. Thus matrix $\mathbf{B}^T\mathbf{CB}$ is singular for kinematically indeterminate (underconstrained) structures including tensegrity systems. This matrix, however, is regular and strictly positive definite for traditional kinematically determinate structures. Steps (2) and (3) of the proof are subtler and worth illustrating by an example given in Fig. 1.

Consider a hexagonal cable net (Fig. 1). This net is initially pre-tensioned by equal forces $P$ directed along the "radiuses". Equilibrium is provided when axial

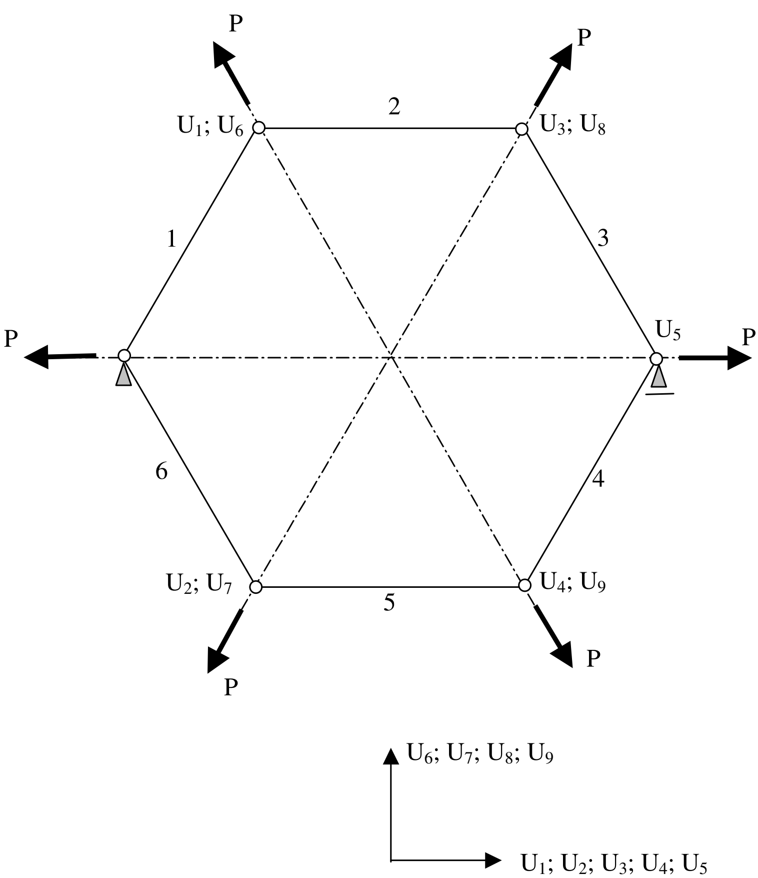



Fig. 1. Cable hexagon. $U_i$ is the nodel degree of freedom. Cables 1–6 are pretensioned due to the diagonal forces $P$.

Table 1. Geometric stiffness matrix $\mathbf{D}$ for the cable hexagon.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\frac{P_1}{L_1}+\frac{P_1}{L_2}$ | | $-\frac{P_2}{L_2}$ | | | | | | |
| | $\frac{P_6}{L_6}+\frac{P_5}{L_5}$ | | $-\frac{P_5}{L_5}$ | | | | | |
| $-\frac{P_2}{L_2}$ | | $\frac{P_2}{L_2}+\frac{P_3}{L_3}$ | | $-\frac{P_3}{L_3}$ | | | | |
| | $-\frac{P_5}{L_5}$ | | $\frac{P_4}{L_4}+\frac{P_5}{L_5}$ | $-\frac{P_4}{L_4}$ | | | | |
| | | $-\frac{P_3}{L_3}$ | $-\frac{P_4}{L_4}$ | $\frac{P_3}{L_3}+\frac{P_4}{L_4}$ | | | | |
| | | | | | $\frac{P_1}{L_1}+\frac{P_2}{L_2}$ | | $-\frac{P_2}{L_2}$ | |
| | | | | | | $\frac{P_6}{L_6}+\frac{P_5}{L_5}$ | | $-\frac{P_5}{L_5}$ |
| | | | | | $-\frac{P_2}{L_2}$ | | $\frac{P_2}{L_2}+\frac{P_3}{L_3}$ | |
| | | | | | | $-\frac{P_5}{L_5}$ | | $\frac{P_4}{L_4}+\frac{P_5}{L_5}$ |

tension forces in all members equal $P$. Thus a nonzero initial equilibrium state exists. Imposing three constraints at supporting points, the number of degrees of freedom is $M = 9$ (only a plane problem is considered). The number of the assembly members is $N = 6$. The 6 by 9 matrix $\mathbf{B}$ posses a nontrivial nullspace and the structure is kinematically indeterminate. Matrix $\mathbf{B}^T\mathbf{CB}$ is singular and positive semi-definite. Geometrical stiffness matrix $\mathbf{D}$ is given in Table 1. Empty cells designate zeros. The values of the initial forces are not specified in order to demonstrate the structure of the matrix, however, all member forces are positive $P_i > 0$, i.e. all members are tensioned. Since member lengths are also positive, the sum of the absolute values of non-diagonal members in any row is not greater than the diagonal entry of the same row. The latter means that matrix $\mathbf{D}$ is at least positive semi-definite. To make the last step and to prove the non-singularity of matrix $\mathbf{D}$, and consequently of $\mathbf{K}$, it is necessary to consider the connectivity and diagonal dominance of matrix $\mathbf{D}$.

The invertability of an $N$ by $N$ matrix $\mathbf{D}$ is achieved because: (a) it is possible to find $g$, $1 \leq g \leq N$ nonintersecting sets of distinct integers (covering all integers from 1 to $N$) among integers from 1 to $N$, so that for every pair of integers $p_t$, $q_t$ of the $t$th set there is a sequence of distinct integers belonging to the same set, $k_1 = p_t, k_2, \ldots, k_{m-1}, k_m = q$, such that all of the matrix entries $D_{k_1k_2}$; $D_{k_2k_3}, \ldots, D_{k_{m-1}k_m}$ are nonzero; (b) for every set defined above it is possible to find at least one integer $s_t$ such that the value of $D_{s_t,s_t}$ is strictly greater than the sum of the absolute values of the non-diagonal row entries. In the considered example, the appropriate sets of numbers are 1, 2, 3, 4, 5 and 6, 8 and 7, 9. The large number of the rows with the diagonal dominance is occasional in the considered example. However, at least one row always exists if one supporting point exists. The connectivity is provided within every set because the graph of the set exhibiting its connectivity is the structure itself (!), i.e. one can "move" from any number (node) to any other number (node) within the set throw the structural members. Consider, for example, a pair of numbers 1 and 5 for the first set. The necessary sequence of nonzero entries is $D_{13}$; $D_{35}$. If the pairs 1 and 2 is taken then the necessary sequence of nonzero entries is $D_{13}$; $D_{35}$; $D_{54}$; $D_{42}$ and so on.

Replace now the initial external "radial" forces $P$ providing tension in cables by the compression forces in nonintersecting struts as shown in Fig. 2. In this case the geometric stiffness matrix $\mathbf{D}$ given in Table 2 includes compression forces. The new terms destroy the diagonal dominance. Substituting $P_i = P$, $i = 1, \ldots, 9$; $L_i = L, i, \ldots, 6$; $L_i = L/2$, $i = 7$, 8, 9 and normalizing with respect to $P/L$ it is possible to obtain matrix $\mathbf{D}$ in numerical form as given in Table 3. Numerical analysis shows that this matrix is positive semi-definite. It is singular with rank 6. On the other hand, matrix $\mathbf{B}^T\mathbf{CB}$ is also singular and positive semi-definite (matrix $\mathbf{B}$ is given in Table 4). Its rank is 8 and its nullspace vector may be written in the form:

$$\mathbf{a} = \{-\sqrt{3}/2; -2/\sqrt{3}; -\sqrt{3}/2; -2/\sqrt{3}; 1/\sqrt{3}; 1/2; -2/3; -5/6; 1\}^T .$$

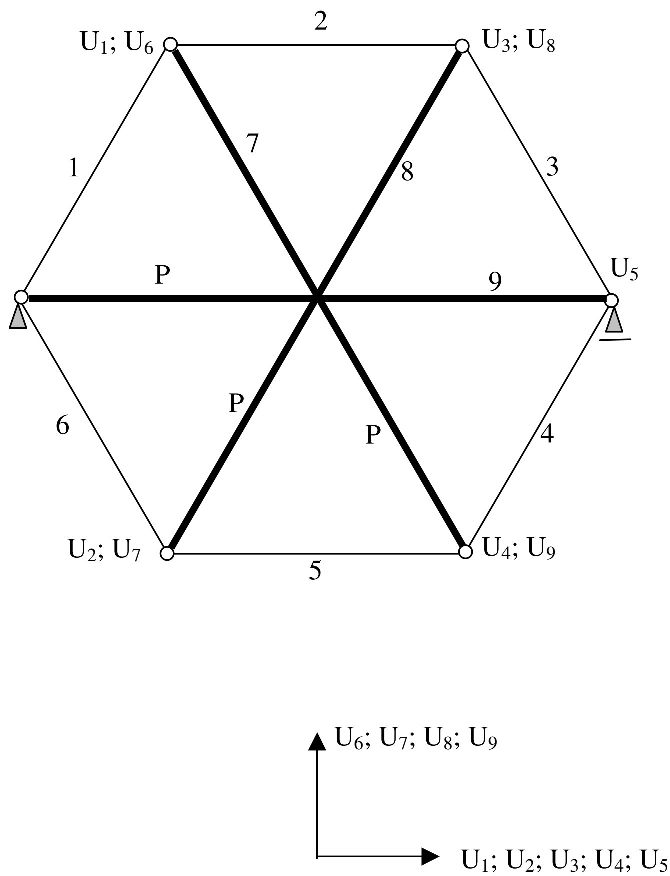



Fig. 2. Tensegrity hexagon. $U_i$ is the nodel degree of freedom. Cables 1–6 are pretensioned due to the compression forces $P$ in struts 7, 8 and 9.

Though both matrix terms forming the tangent stiffness are positively semi-definite and singular, the tangent stiffness itself is strictly positive definite. This may be checked by multiplication: $\mathbf{Da} \neq \mathbf{0}$. Its result is a nonzer vector. The latter means that the nullspaces of $\mathbf{B}^T\mathbf{CB}$ and $\mathbf{D}$ are different, and, consequently $\mathbf{K}$ is strictly positive definite. It is remarkable that the material properties of members were not involved in the previous reasoning and the stability property is purely statical/kinematical as in the case of cable nets.

The formal difference between the considered examples is that in the case of the cable hexagon the external nodal loads are "dead", while in case of the tensegrity hexagon the "internal nodal loads" — compressed struts are not "dead", they are "following loads". The latter means the necessity to modify the tangent stiffness matrix. It seems, however, that being formally different both cases are physically equivalent as the Figs. 1 and 2 prompt. Considering any tensegrity system, one can "instantaneously replace" compressed struts by "equivalent" external nodal loads

Table 2. Geometric stiffness matrix $\mathbf{D}$ for the tensegrity hexagon.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\frac{P_1}{L_1}+\frac{P_2}{L_2}-\frac{P_7}{L_7}$ | | $-\frac{P_2}{L_2}$ | $\frac{P_7}{L_7}$ | | | | | |
| | $\frac{P_6}{L_6}+\frac{P_5}{L_5}-\frac{P_8}{L_8}$ | $\frac{P_8}{L_8}$ | $-\frac{P_5}{L_5}$ | | | | | |
| $-\frac{P_2}{L_2}$ | $\frac{P_8}{L_8}$ | $\frac{P_2}{L_2}+\frac{P_3}{L_3}-\frac{P_8}{L_8}$ | | $-\frac{P_3}{L_3}$ | | | | |
| $\frac{P_7}{L_7}$ | $-\frac{P_5}{L_5}$ | | $\frac{P_4}{L_4}+\frac{P_5}{L_5}-\frac{P_7}{L_7}$ | $-\frac{P_4}{L_4}$ | | | | |
| | | $-\frac{P_3}{L_3}$ | $-\frac{P_4}{L_4}$ | $\frac{P_3}{L_3}+\frac{P_4}{L_4}-\frac{P_9}{L_9}$ | | | | |
| | | | | | $\frac{P_1}{L_1}+\frac{P_2}{L_2}-\frac{P_7}{L_7}$ | | $-\frac{P_2}{L_2}$ | $\frac{P_7}{L_7}$ |
| | | | | | | $\frac{P_6}{L_6}+\frac{P_5}{L_5}-\frac{P_8}{L_8}$ | $\frac{P_8}{L_8}$ | $-\frac{P_5}{L_5}$ |
| | | | | | $-\frac{P_2}{L_2}$ | $\frac{P_8}{L_8}$ | $\frac{P_2}{L_2}+\frac{P_3}{L_3}-\frac{P_8}{L_8}$ | |
| | | | | | $\frac{P_7}{L_7}$ | $-\frac{P_5}{L_5}$ | | $\frac{P_4}{L_4}+\frac{P_5}{L_5}-\frac{P_7}{L_7}$ |

Table 3. Geometric stiffness matrix **D** for the tensegrity hexagon normalized by $P/L$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3/2 | | −1 | 1/2 | | | | | |
| | 3/2 | 1/2 | −1 | | | | | |
| −1 | 1/2 | 3/2 | | −1 | | | | |
| 1/2 | −1 | | 3/2 | −1 | | | | |
| | | −1 | −1 | 3/2 | | | | |
| | | | | | 3/2 | | −1 | 1/2 |
| | | | | | | 3/2 | 1/2 | −1 |
| | | | | | −1 | 1/2 | 3/2 | |
| | | | | | 1/2 | −1 | | 3/2 |

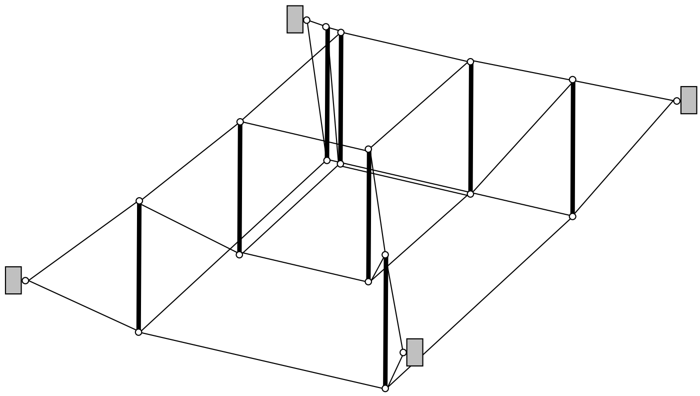
Table 4. Matrix of direction consines **B** for the tensegrity hexagon.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $-1/2$ | | | | | $-\sqrt{3}/2$ | | | |
| $1$ | | $-1$ | | | | | | |
| | | $1/2$ | | $-1/2$ | | | $-\sqrt{3}/2$ | |
| | | | $1/2$ | $-1/2$ | | | | $\sqrt{3}/2$ |
| | $1$ | | $-1$ | | | | | |
| | $-1/2$ | | | | | $\sqrt{3}/2$ | | |
| $-1/2$ | | | | | $\sqrt{3}/2$ | | | $-\sqrt{3}/2$ |
| | $-1/2$ | $1/2$ | | | | $-\sqrt{3}/2$ | $\sqrt{3}/2$ | |
| | | | $1/2$ | $1$ | | | | |

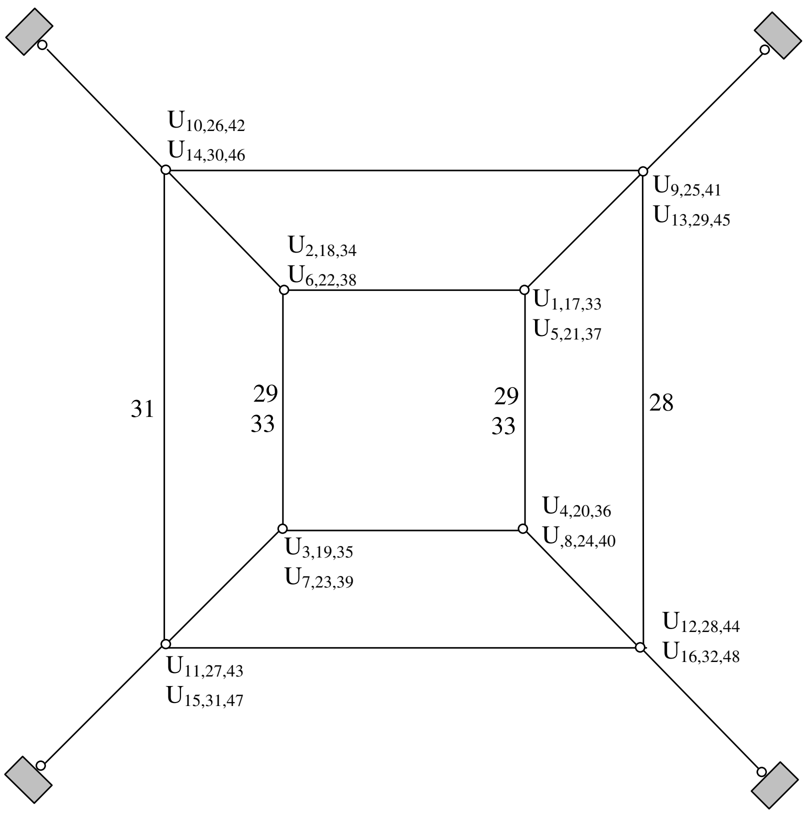
and to analyze the remaining cable system. Such cable system is always stable, so the whole tensegrity system can be conjectured to be stable.

> *Conjecture*: *Any tensegrity system with totally tensioned cables is stable independently of its topology, geometry and specific magnitudes of the member forces.*

The considered above example presents the plane tensegrity structure. We also analyzed a space tensegrity structure (Fig. 3). It is a cut-down version of the famous Geigers dome comprising 8 vertical struts and 28 cables. It possesses 48 degrees of freedom. Its degree of kinematic indeterminacy is 13. Pre-stressing forces take the following dimensionless values accounting for symmetry: $P_1 = P_2 = 3.464P$, $P_3 = P_4 = 1.732P$, $-P_5 = P_{30} = P_{32} = P$, $P_{31} = -P_{34} = 2P$. Omitting the intermediate results it was obtained that this structure is also stable for arbitrary $P > 0$. The two considered examples may be completed with other stable tensegrities reported by Coughlin and Stamenovich,[7] Murakami,[22] Oppenheim and Williams,[23] Sultan *et al.*,[27] Vassart *et al.*,[31] Vilnay,[32] Volokh,[34] Wang,[40] Yuan and Dong,[41] and the references therein.

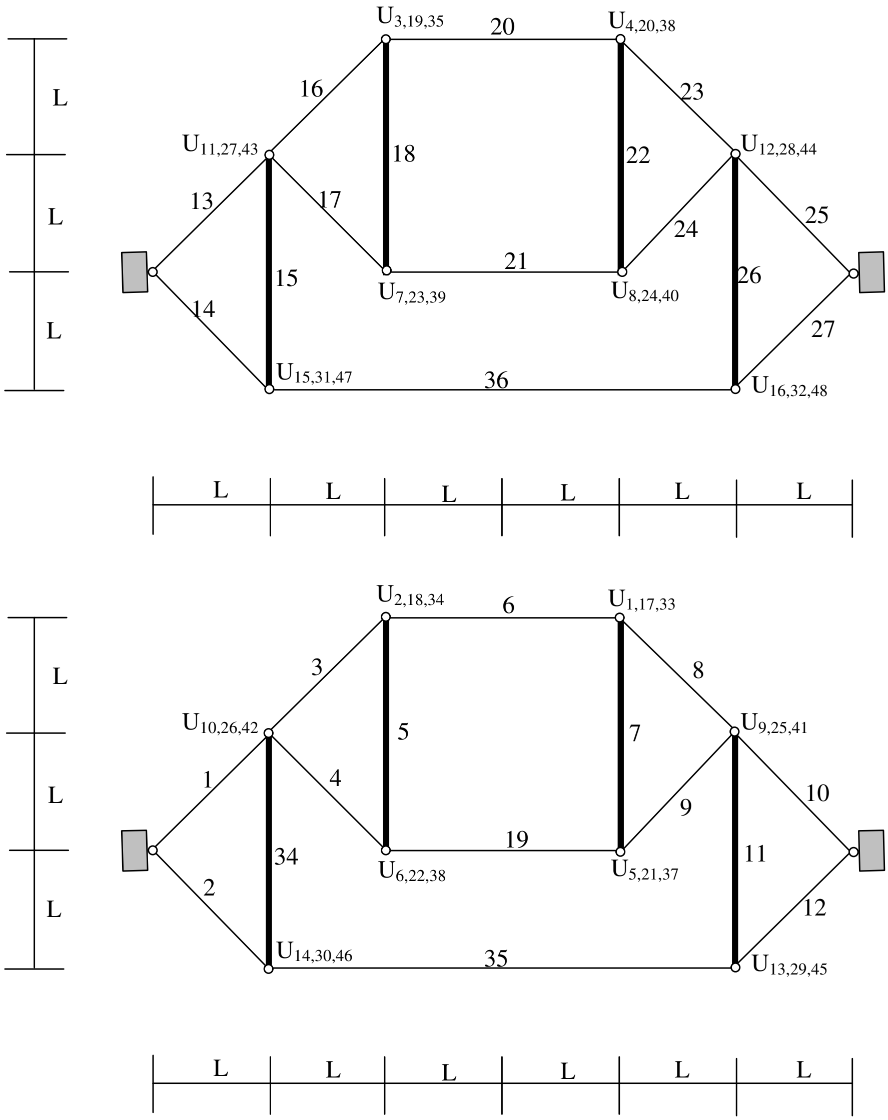


(a)




(b)

Fig. 3. (a) Geiger's tensegrity dome. 3D view. Vertical bold lines designate compressed struts. Fine lines are tensioned cables. (b) Geiger's tensegrity dome. Top view. Subscripts designate nodal degrees of freedom in three perpendicular directions. (c) Geiger's tensegrity dome. Front view. Subscripts designate nodal degrees of freedom in three perpendicular directions.




(c)

Fig. 3. (*Continued*)

Returning to works by Connelly and Back[5] and Connelly and Whiteley[6] it is interesting to note that these authors found that cables-struts assemblies are not necessarily stable if more than *one* strut is attached at a node. This finding is in agreement with our conjecture since only structures with one strut per node are defined as tensegrities here. On the other hand this finding sharpens our conjecture and excludes its generalization.

## 4. Conclusions

Stability of tensegrity structures, which are tensioned cable nets prestressed by isolated compressed struts, has been considered. General formulation of the stability problem for tensegrity structures is based on the direct use of global nodal degrees of freedom and the strain energy expression. This setting is more straightforward than the use of the standard finite element procedures for the geometrically nonlinear truss formulation where tedious preparations, including local element formulation, transition to global formulation and assembling of elements, are required. Based on this formulation and the established result on the stability of pretensioned cable nets, the stability of tensegrity structures is conjectured. It is particularly conjectured that a tensegrity structure with all tensioned cables and compressed struts is always stable independently of its topology, geometry and specific magnitudes of member forces. This conjecture is illustrated by examples of space and plane tensegreties. It is also justified my numerous examples of the stability of tensegrity structures considered by other authors. It is hoped that a possible proof of the stability conjecture will allow for deeper insights in the general problem of stability of trusses. Such proof will also abandon analysis of the stability of any tensegriy in both initial and loaded states if all cables are tensioned.

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