Spherical void expansion in rubber-like materials: The stabilizing effects of viscosity and inertia

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A R T I C L E   I N F O

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A B S T R A C T

Dynamic cavitation is known to be a typical failure mechanism in rubber-like solids. While the mechanical behaviour of these materials is generally rate-dependent, the number of theoretical and numerical works addressing the problem of cavitation using nonlinear viscoelastic constitutive models is scarce. It has been only in recent years when some authors have suggested that cavitation in rubber-like materials is a dynamic fracture process strongly affected by the rate-dependent behaviour of the material because of the large strains and strain rates that develop near the cavity. In the present work we further investigate previous idea and perform finite element simulations to model the dynamic expansion of a spherical cavity embedded into a rubber-like ball and subjected to internal pressure. To describe the mechanical behaviour of the rubber-like material we have used an experimentally calibrated constitutive model which includes rate-dependent effects and material failure. The numerical results demonstrate that inertia and viscosity play a fundamental role in the cavitation process since they stabilize the material behaviour and thus delay failure.

1. Introduction

An isolated cavity/void inside a solid, subjected to load either at the cavity wall or at the remote field, expands rapidly after a critical load is reached. This phenomenon is referred to as cavitation instability. Cavitation is known to be a typical failure mechanism in solids. Experimental evidences of cavitation and fracture led by cavitation are available for different materials such as ductile metals, polymers, elastomers and biological tissues [4,17,16,20,28,11,47]. When the applied load (either at the cavity wall or at the remote field) is less than the critical load for cavitation, a new equilibrium configuration is reached after some expansion of the void. When the applied load is higher than the critical load, the void expands at finite velocity and ultimately leads to material failure. If the expansion velocity is high enough, dynamic effects become meaningful and the void growth is influenced by inertia. A large number of analytical and numerical studies have been devoted to the analysis of quasi-static and dynamic cavitation in a wide variety of materials [26,12,6,5,2,25,24,35,23,13,41,18,46,14,45,21,9,39,40,37,7].

With growing applications of soft and biological materials, the study of cavitation phenomena in elastic solids has become increasingly important. In particular, the acclaimed review articles published by Gent [19], Horgan and Polignone [23] and Fond [18] attracted the attention of the Solid Mechanics community to the problem of cavitation in rubber-like materials. More recently, one should highlight the work of López-Pamies et al. [33,34] who introduced a new theory to model static cavitation in elastomeric solids that considers general 3D loading conditions and incorporates direct information of the underlying defects at which cavitation can initiate. Just one year ago, López-Pamies and co-workers [29] pushed forward the cavitation problem in elastomers and showed the need of including damage/failure mechanisms in the analysis because rubber-like materials fail at large, but finite strains. A similar idea was developed by Lev and Volokh [31] who used a constitutive model which incorporates failure for analyzing cavitation in rubber. Using various nonlinear elastic material models, Lev and Volokh [31] demonstrated the interplay between elasticity and fracture in the development of the cavitation process.

Though most rubber-like materials are rate-dependent, there are not many theoretical and numerical works which consider viscoelastic constitutive models to analyze the dynamic cavitation problem. Last year, Cohen and Molinari [10] presented a theoretical framework to investigate dynamic cavitation in viscoelastic incompressible materials modelled with an hereditary integral-type formulation. To facilitate analytical solutions, Cohen and Molinari [10] considered two specific loading cases, namely, a sudden constant deformation and a deformation that increases at constant rate. Their objective was to provide...
closed-form expressions to measure the local viscoelastic properties of the rubber-like material through controlled relaxation experiments. Moreover, Kumar et al. [27] just published a paper that presents new insights into the relevance of inertial and viscous dissipation effects on the onset of cavitation in rubber. They concluded that viscosity and dynamic effects increase the values of the applied loads at which cavitation occurs. In the present work, we further investigate previous ideas and approach, using finite element simulations in ABAQUS/Explicit, the canonical problem of a spherical void embedded at the center of an elastic ball and subjected to internal pressure. The elastic medium is modelled with a rate-dependent constitutive model which accounts for material failure using energy limiters [42,44,3]. The finite element calculations confirm the results of Cohen and Molinari [10] and Kumar et al. [27] and show, systematically, the stabilizing effect of viscosity and inertia.

An outline of the paper is as follows. Section 2 records the basic equations of a viscoelastic model which describes the rate-dependent response of Styrene Butadiene rubber within a wide range of strain rates. Section 3 reviews a classical analytical solution for the dynamic cavitation of a spherical void embedded into an incompressible elastic ball and subjected to internal pressure. Section 4 presents a finite element model developed in ABAQUS/Explicit to simulate dynamic expansion of a spherical cavity inserted in a slightly compressible viscoelastic ball. The finite element results are presented in Section 5 and rationalized with the predictions of the theoretical model. The main conclusions of this research are presented in Section 6.

2. Nonlinear viscoelasticity with energy limiters

In this section we summarize the formulation of nonlinear viscoelasticity with energy limiters developed by Aranda-Iglesias et al. [3] which adapted the Eulerian constitutive framework for large inelastic deformations previously proposed by Volokh [43].

2.1. Basic equations

Consider a material point that occupies position $X$ in the reference configuration $X_0$ of a deformable body. The current position vector $x$ in the deformed configuration $X$ is given by $x = \chi(X, t)$, where $\chi$ is a bijective and twice continuously differentiable mapping. Deformation in the vicinity of the material point is described by the deformation gradient tensor $F$ and angular velocity $\omega$

$$F = \frac{\partial x}{\partial X},$$

(1)

where $x(t)$ is the velocity vector and $x_0(t)$ and $v_0$ are prescribed in $A$.

$$\sigma = \sigma^T,$$

(3)

where the divergence operator is calculated with respect to the current coordinates $\chi$; $\sigma$ is the Cauchy stress tensor; $b$ is the body force per unit of current volume; $\rho$ and $a$ are the current mass density and acceleration vector, respectively.

The balance of linear momentum on the body surface $\partial\Omega$ reads

$$\sigma n = f,$$

(4)

where $f$ is a prescribed traction on unit area of the surface with the unit outward normal $n$.

Alternatively to (4), a surface boundary condition can be imposed on placements

$$x = \chi,$$

(5)

where the barred quantity is prescribed on the surface $\partial\Omega$.

The initial conditions are

$$x(t = 0) = x_0, \quad v(t = 0) = v_0,$$

(6)

where $v$ is the velocity vector and $x_0$ and $v_0$ are prescribed in $A$.

where $\psi$ is the strain energy function of spring A

$$\psi^A = H(\xi)\psi^A, \quad \psi^A = \psi^A(1),$$

(9)

$$\psi^B = \psi^B(\xi) \mathbf{s}^B \mathbf{s}^B \rightarrow \infty \Rightarrow \psi^B = 0,$$

(10)

where $\psi^A$ and $\psi^B$ are the strain energy function of spring A which serves to characterize the thermodynamic equilibrium states of the elastomers $\psi^A$ and $\psi^B$ is the strain energy function of spring B which serves to account for the additional energy storage and non-equilibrium states. Furthermore, $B = \mathbf{s}^B$ is the unit Cauchy-Green strains tensor, $\mathbf{s}^B$ is an (strain like) internal variable of the model and $\xi$ is a switch parameter (that will be defined later). We further impose the following conditions on the spring energy function of spring A

$$\psi^A = \psi^A(1),$$

(8)

$$\psi^B = \psi^B(1),$$

(9)

$$\psi^B = \psi^B(1),$$

(10)

where $\psi^A$, $\psi^B$, and $\psi^C$ designate the constant bulk failure energy and the elastic free energy of spring A, respectively. Moreover, $H(\xi)$ is a unit step function, i.e., $H(\xi) = 0$ if $\xi < 0$ and $H(\xi) = 1$ otherwise; and $\mathbf{I}$ is a second-order identity tensor; and $\| \mathbf{B} \|$ is a tensor norm.

The switch parameter $\xi \in (-\infty, 0]$ is defined by the evolution equation

$$\dot{\xi} = -H(\xi - \psi^A) \psi^B, \quad \xi(t = 0) = 0,$$

(11)

where $0 < \xi < 1$ is a dimensionless precision constant. Note that a superposed dot denotes differentiation with respect to time.

The physical interpretation of the strain energy function is straightforward: the response of spring A is elastic as long as the strain energy is below its limit, $\psi^A$. When the limit is reached, the strain energy remains constant for the rest of the deformation process, thereby making material healing impossible. The parameter $\xi$ is not an internal variable; it works as a switch: if $\xi = 0$ then the process is elastic and if $\xi < 0$ then the material is irreversibly damaged and the stored energy is dissipated.

In order to enforce the energy limiter in the strain energy function, we use the following form of the elastic energy

$$\psi^A(\mathbf{B}) = \frac{\Phi}{m} \frac{1}{m} \mathbf{W}_\Lambda(\mathbf{B}^m),$$

(12)

where $\Gamma(x,t) = \int_t^\infty e^{-\xi} \mathbf{W}_\Lambda(\mathbf{B}^m)$ is the upper incomplete gamma function, $\mathbf{W}_\Lambda(\mathbf{B})$ is the strain energy function of intact material, $\Phi$ is the energy limiter and $m$ is a dimensionless material parameter which controls the sharpness of the transition to material failure in the stress-strain curve. Increasing or decreasing $m$ is possible to simulate more or less steep ruptures of the internal bonds accordingly.

The failure energy can be calculated as follows
ψ_β = \frac{\Phi f}{m} \left( \frac{1}{m} \cdot \frac{W_i (1)^n}{\theta^m} \right).

(13)

Note that the failure energy is a constant that depends on the two failure parameters (Φ, m) through the gamma function. There is no need to limit the energy of spring B as long as the failure of spring A leads to overall failure. Therefore, we define the strain energy function for spring B as

ψ_β (B_β, ε) = H (ε) W_β (B_β).

(14)

where W_β(B_β) stands for the strain energy without failure. Note that this formulation is valid for any pair of strain energies W_α and W_β used to describe the intact behaviour of the material (see Section 2.4).

Based on the additive decomposition of the strain energy function ψ, the Cauchy stress is given by

σ = σ_α + σ_β,

(15)

where

σ_α = 2I^{−1/2}_3 \frac{\partial \psi_α}{\partial \B_α} \B = 2I^{−1/2}_3 (I_3 \psi_α \B + \psi_β \B^2),

(16)

σ_β = 2I^{−1/2}_3 \frac{\partial \psi_β}{\partial \B_β} \B_β = 2I^{−1/2}_3 (I_3 \psi_β \B + \psi_β \B^2).

(17)

The principal invariants are

I_1 = tr \B,
I_2 = (tr \B)^2 - tr (\B^2),
I_3 = det \B,

(18)

I_1 = tr \B_β,
I_2 = (tr \B_β)^2 - tr (\B_β^2),
I_3 = det \B_β,

(19)

where ψ_β = ψ_β/∂I_1 and ψ_β = ψ_β/∂I_3.

The constitutive law (flow rule) for the dashpot is written in the following general form

β_B = \beta_1 + \beta_2 D_β + \beta_3 D_β^2,

(20)

where β_1 are function(al)s generally, depending on stresses and strains and D_β is the rate of deformation tensor corresponding to the dashpot. Note that, as demonstrated by Aranda-Iglesias et al. [3], the second law of thermodynamics requires that the following relation is fulfilled

β_1 D_β + β_2 D_β^2 + β_3 D_β^3 ≥ 0.

(21)

Following Eckart [15], Leonov [30] and Volokh [43] the relation between B_β and D_β can be written as follows

∇ B_β + D_β D_β + B_β D_β = 0.

(22)

where

∇ B_β = B_β - LB_β - B_β L^T.

(23)

is the Oldrody objective rate of the (strain like) internal variable B_β. In the previous expression L refers to the velocity gradient tensor of the whole model.

2.3. Specialization to incompressible and compressible materials

In this section we specialize the constitutive model to incompressible and (slightly) compressible materials. The hypothesis of incompressibility is used in Section 3 to show the analytical (classical) solution of the dynamic (spherical) cavitation problem. Compressibility of the material is taken into account in the numerical model presented in Section 4.

2.3.1. Incompressible formulation

The incompressibility condition implies that det \B = 1, det B_β = 1 and tr D_β = 0. With these constraints, the constitutive laws for the springs (16)-(17) are written as follows

σ_α = -p_α I + 2(ψ_α + I \psi_β) \B - ψ_β \B^2,

(24)

σ_β = -p_β I + 2(ψ_β + I \psi_β) \B_β - ψ_β \B_β^2,

(25)

where p_α and p_β are undefined Lagrange multipliers enforcing incompressibility.

The constitutive law for the dashpot (20) can be written as

β_1 = \frac{1}{3} tr σ_α,
β_2 = η_2, β_3 = 0,

(26)

where η_2 is the only viscosity parameter (or function).

Substitution of (26) in (20) yields

σ_α = \frac{1}{3} (tr σ_α) \B + η_2 D_β.

(27)

Moreover, the thermodynamic restriction requires that

η_2 ≥ 0.

(28)

2.3.2. Compressible formulation

Implementation of strict incompressibility is difficult (and unnecessary) in numerical simulations. Hence the constitutive equations are modified by penalizing the incompressibility conditions with large bulk moduli to implement material compressibility. The constitutive laws for the springs are written as

σ_α = 2I^{−1/2}_3 ((I_3 k_1 - k_2) \B + (\psi_α + I \psi_β) \B - ψ_β \B^2),

(29)

σ_β = 2I^{−1/2}_3 ((I_3 k_1 - k_2) \B_β + (\psi_β + I \psi_β) \B_β - ψ_β \B_β^2),

(30)

where k_1, k_2 and k_1B, k_2B are the penalizing bulk moduli for springs A and B accordingly.

We note that the bulk moduli are not independent and they should obey the condition of zero stress for B = B_β = I and D = 0. Thus, we obtain the relations

k_1 ≈ k_2 ≥ 0, k_1B ≈ k_2B ≥ 0,

(31)

which allow to obtain slight compressibility in the numerical simulations.

The constitutive law for dashpot is specified in a simple form to reproduce the flow rule proposed by Reese and Govindjee [36] as follows,

β_1 = \frac{3η_1 - 2η_2}{9η_1} tr σ_α,
β_2 = η_2, β_3 = 0,

(32)

where η_1, η_2 are two viscosity parameters (or functions). One of the major advantages of this flow rule is that it takes into account for the large deformations and hence it is suitable for rubber-like materials.

Substituting (32) in (20) yields

σ_α = \frac{3η_1 - 2η_2}{9η_1} (tr σ_α) \B + η_2 D_β.

(33)

Moreover, the thermodynamic restriction requires that

η_2 ≥ 0, 3η_1 ≥ 2η_2.

(34)

2.4. Material parameters

Following Aranda-Iglesias et al. [3], we use the formulation proposed by López-Pamies [32] for the intact strain energy functions

W_α (B) = \frac{\psi_α^{(1-n_1)}}{2n_1} \mu_1 (I^n - 3I^n) + \frac{\psi_α^{(3-n_1)}}{2n_1} \mu_2 (I^{2n_1} - 3I^{2n_1}),

(35)

and

W_β (B_β) = \frac{\psi_β^{(1-n_2)}}{2n_2} \mu_1 (I_β^{2n_1} - 3I_β^{2n_1}) + \frac{\psi_β^{(3-n_2)}}{2n_2} \mu_2 (I_β^{2n_1} - 3I_β^{2n_1}),

(36)
The viscosity function is taken from Hoo Fatt and Ouyang [22] and reads as follows

\[ \eta_c = (C_1(1 - \exp(C_2((I - 3)))) + C_3)(C_4 B_1 + C_5 B_2 + C_6 B_3 + C_7). \]  

(37)

The complete constitutive model has 17 parameters: 6 for spring A \((\mu_1, \mu_2, \alpha_1, \alpha_2, m, \Phi)\), 4 for spring B \((\mu_{B1}, \mu_{B2}, \alpha_{B1}, \alpha_{B2})\) and 7 for the dashpot \((C_1, C_2, C_3, C_4, C_5, C_6, C_7)\). Aranda-Iglesias et al. [3] calibrated the constitutive model to describe the mechanical behaviour of Styrene Butadiene rubber. Small particles of Styrene Butadiene rubber are added to a large number of semi-crystalline brittle polymers to increase their toughness ([8]). In these rubber-modified polymers major source of toughening comes from the energy consumed by the deformation of rubber particles before they fail due to cavitation ([38]). Due to higher toughness most of these polymers are used in impact applications. Thus study of cavitation in Styrene Butadiene rubber is important.

The values of the parameters are listed in Table 1. Moreover, the bulk moduli \((k_1, k_2, k_{B1}, k_{B2})\) take the value \(10^{11}\) MPa for all the numerical simulations reported in this paper. The material density is \(900\) kg/m\(^3\).

Note that, in agreement with the experimental evidence reported by Hoo Fatt and Ouyang [22], the constitutive model captures the rate-independent response of the material at sufficiently low and high strain rates. In the quasi-static limit and for strain rates above \(\approx 2000\) s\(^{-1}\) the branch B of the rheological model will exhibit purely elastic response.

### Table 1

| Material parameters for Styrene Butadiene rubber as taken from Aranda-Iglesias et al. [3]. |
|----------------------------------|----------------|----------------|--------------------|--------|----------------|
| \(\mu_1\) (MPa)          | \(a_1\)       | \(\mu_2\) (MPa) | \(a_2\)          | \(m\)  | \(\Phi\) (MPa) |
| 0.391                        | 1.045         | 2.162          | -3.065           | 30     | 7.5            |

<table>
<thead>
<tr>
<th>Spring B</th>
<th></th>
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<td>(\mu_{B1}) (MPa)</td>
<td>(\alpha_{B1})</td>
<td>(\mu_{B2}) (MPa)</td>
<td>(\alpha_{B2})</td>
<td></td>
</tr>
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<td>2.868</td>
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<table>
<thead>
<tr>
<th>Dashpot B</th>
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</thead>
<tbody>
<tr>
<td>(C_1) (MPa s)</td>
<td>(C_2)</td>
<td>(C_3) (MPa s)</td>
<td>(C_4)</td>
<td>(C_5)</td>
</tr>
<tr>
<td>23.095</td>
<td>7.421 \times 10^{-4}</td>
<td>-8.458 \times 10^{-2}</td>
<td>-872.52</td>
<td>-7975.595</td>
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</tbody>
</table>

3. Theoretical model

We present the main features of the analytical (classical) solution for the dynamic cavitation of a spherical void embedded in an incompressible elastic ball and subjected to internal pressure. Note that the material is taken as purely elastic, i.e. viscous effects are not accounted for. The analytical solution will be used in Section 5 as a reference to rationalize the finite element results presented in Section 5.

3.1. Radially symmetric dynamic deformations

If the material is deformed so that the spherical symmetry is maintained, the motion is given by

\[ r = r(R,t), \quad \theta = \Theta, \quad \omega = \Omega, \]  

(38)

where \((r, \theta, \omega)\) denote the current coordinates of a point having coordinates \((R, \Theta, \Omega)\) in the undeformed configuration. For the motion described in (38), the deformation gradient tensor is

\[ F = \frac{dr}{dR} e_\theta \otimes e_R + \frac{r}{R} e_\theta \otimes e_R + \frac{r}{R} e_\omega \otimes e_R, \]  

(39)

where \((e_R, e_\theta, e_\omega)\) and \((e_R, e_\theta, e_\omega)\) are reference and current base vectors in standard spherical coordinate system, respectively. Moreover, the displacement of a material point is

\[ u = u_r(r, t) e_r, \]  

(40)
response of springs A and B. Next, for the quasi-static case, we derive theoretical solutions for the lower and upper bounds that will be used to rationalize the finite element simulations reported in Section 5.

- **Lower bound solution**
  Considering only spring A, the strain energy function becomes
  \[\psi(B) = \psi_A(B, \xi),\]
  which yields
  \[\psi_1 = H(\xi) \exp \left( -\frac{W^m}{\phi^m} \frac{\partial W_A}{\partial I_b} \right)\]
  (49)
  \[\psi = H(\xi) \exp \left( -\frac{W^m}{\phi^m} \frac{\partial W_A}{\partial I_b} \right)\]
  (50)

- **Upper bound solution**
  Considering springs A and B, the strain energy function becomes
  \[\psi(B) = \psi_A(B, \xi) + \psi_B(B, \xi),\]
  which yields
  \[\psi_1 = H(\xi) \exp \left( -\frac{W^m}{\phi^m} \frac{\partial W_A}{\partial I_b} + \frac{\partial W_B}{\partial I_b} \right)\]
  (51)
  Using the definitions of \(\sigma_a\) and \(\sigma_{bb}\) given in (24) we obtain
  \[\sigma_a - \sigma_{bb} = 2\psi_1(B_b - B_{bb}).\]
  (52)
  Inserting (52) into (45) and neglecting the inertia term we obtain
  \[P(t) = -\int_r^b 4\psi_1(I_b(r)) \left( \frac{R^4}{r^4} - \frac{r^4}{R^4} \right) \frac{dr}{r}.\]
  (53)
  with
  \[I_b(r) = \frac{R^4}{r^4} + 2\frac{r^4}{R^4}\]
  and
  \[R(r, a) = (r^3 - a^3 + a_0^{1/3})^{1/3}.\]
  (54)
  Replacing \(\psi_1\) in (53) by (49) and (51) we obtain the lower bound \(P_{\text{lower}}\) and upper bound \(P_{\text{upper}}\) solutions for the static pressure, respectively. These are shown in Fig. 2 as a function of the normalized void radius \(a/a_0\). We observe that \(P_{\text{lower}}\) and \(P_{\text{upper}}\) increase rapidly (and almost linearly) for short values of \(a/a_0\) and saturate for large values of \(a/a_0\). Within the whole range of values of \(a/a_0\) explored, \(P_{\text{lower}}\) goes below \(P_{\text{upper}}\) as expected. Critical pressures corresponding to unstable expansion of void are obtained as \(P_{\text{lower}}^{cr} \approx 2\text{ MPa}\) and \(P_{\text{upper}}^{cr} \approx 9.5\text{ MPa}\). Failure at the cavity surface \((r = a)\) will occur when material failure criteria discussed in Section 2 is satisfied. Note that, to obtain the solutions plotted in Fig. 2 we have used \(b_0 = 1000a_0\). This ratio between the inner and outer radius of the elastic ball is large enough to make the solution virtually independent of \(b_0\) ([42]).

4. Numerical model

This section describes the features of the axisymmetric finite element model developed to simulate dynamic spherical cavity expansion. The numerical analyses are carried out using the finite element program ABAQUS/Explicit ([1]). The problem setting is of a very large sphere of radius \(b_0 = 100\text{ mm}\) with a small cavity in its center of radius \(a_0\). Three values of \(a_0\) are explored in the numerical calculations presented in Section 5, namely 0.01, 0.1 and 1 mm. Due to the symmetry of the model, only the \(\theta \geq 0\) half of the specimen has been analyzed (see Fig. 3). The solid is initially at rest, stress and strain free, and a pressure \(P(t)\) is applied at the cavity wall.

The model has been meshed using a total of 15,000 four-node axisymmetric reduced integration elements, CAX4R in ABAQUS notation. This number of elements results from placing 50 elements along the circumferential direction and 300 along the radial direction. The mesh shows radial symmetry in an attempt to retain the symmetry of the problem and minimize the potential interference of the mesh on the calculations. The elements size is constant along the circumferential direction whereas it decreases along the radial direction as the cavity is approached. Namely, the dimensions of the elements located at the cavity wall are: 0.3 \(\mu\text{m} \times 0.23 \mu\text{m}\) for \(a_0 = 0.01\text{ mm}\), 3 \(\mu\text{m} \times 23 \mu\text{m}\) for \(a_0 = 0.1\text{ mm}\) and 31 \(\mu\text{m} \times 252 \mu\text{m}\) for \(a_0 = 1\text{ mm}\). Small elements are required to capture the high gradients of stress and strain which arise close to the cavity. A mesh convergence study has been performed, and the time evolution of different critical output variables, namely stress, strain and cavitation velocity were compared against a measure of mesh density until the results converged satisfactorily.

The set of constitutive equations describing the material behaviour presented in Section 2 are implemented into the finite element code through a user subroutine. For that task, we have used the numerical scheme developed by Aranda-Iglesias et al. [3].

5. Results and discussion

In this section we present finite element results for the spherical void expansion problem using two different loading conditions: (1) monotonically increasing pressure and (2) constant pressure. Both loading conditions, widely used in the numerical analysis of cavitation problems ([37,27]), show clear evidences of the stabilizing effect played by viscosity and inertia in the expansion of the void. In selected
cases, the numerical results are further rationalized with the predictions of the theoretical model described in Section 3.

5.1. Loading condition 1: monotonically increasing pressure

A pressure is applied at the cavity wall at constant rate ($P = \text{constant}$). According to Wu et al. [46], different pressure rates represent loading conditions of different severity/intensity (impulsive loading, impact loading, shock loading…). Fig. 4 shows the applied pressure $P$ versus the normalized void radius $a/a_0$ for different values of $P$. Namely, Fig. 4(a) shows data for the purely elastic model and Fig. 4(b) shows results for the (full) viscoelastic constitutive model. In any case, the failure of the material is taken into account using the energy limiter in spring $A$. The first occurrence of failure at the cavity wall is identified with * in the $P - a/a_0$ curves. The initial cavity size is $a_0 = 0.1 \text{mm}$. The theoretical solutions for the pressures corresponding to the lower and upper non-viscous quasi-static bounds ($P_{\text{lower}}$ and $P_{\text{upper}}$, see Section 3.2) are also plotted. For the purely elastic material, since only spring $A$ is considered, we have that $P_{\text{lower}} = P_{\text{upper}}$.

Firstly, we pay attention to Fig. 4(a). For applied pressure rates $P < 10^5 \text{MPa/s}$, the $P - a/a_0$ curves coincide with $P_{\text{lower}} = P_{\text{upper}}$, as inertia effects play a negligible role in the void expansion process. The failure occurs at $a/a_0 \approx 1.6$, when applied pressure becomes higher than $P_{\text{lower}} \approx 2 \text{MPa}$. For applied pressure rates $P > 10^5 \text{MPa/s}$ the $P - a/a_0$ curves deviate from $P_{\text{lower}} = P_{\text{upper}}$. The cavity wall withstands a pressure larger than $P_{\text{lower}}$ due to the contribution of inertia effects. The difference in pressure increases with $a/a_0$. Moreover, the slope of the $P - a/a_0$ curves increases with $P$, which reveals the stabilizing effect of inertia. Note that a $P - a/a_0$ curve with slope tending to 0 represents an unstable growth of the void. The failure occurs for $a/a_0 \approx 4.3$. This value is much larger than the one corresponding to $P < 10^5 \text{MPa/s}$. It becomes apparent that inertia effects delay failure and improve the energy absorption capacity of the material. This is a key result of our investigation which helps to understand the performance of rubber-like materials under impact loading.

Secondly, we focus on Fig. 4(b). For applied pressure rates $P \leq 10^5 \text{MPa/s}$ the $P - a/a_0$ curves lie within the lower and upper bound solutions. Within this range of pressure rates, the increasing pressure at the cavity wall with $P$ is mostly caused by the effect of viscosity. Inertia effects seem to have a secondary contribution to the cavity expansion process. Note that $P_{\text{upper}}$ does not correspond to a possible equilibrium solution but to the maximum load that can be carried by the cavity without intervention of inertia effects. Viscosity impedes to the cavity to reach an equilibrium condition for applied pressures larger than $P_{\text{lower}}$. The cavity radius at the time of failure $(a/a_0 \approx 4.3)$ is much larger than for the rate-independent material $(a/a_0 \approx 1.6)$. It becomes apparent that, in absence of meaningful inertia effects, viscosity delays failure and improves the energy absorption capacity of the material. For applied pressure rates $P > 10^5 \text{MPa/s}$, the $P - a/a_0$ curves run above the upper bound $P_{\text{upper}}$. The viscoelastic material withstands cavity pressures larger than $P_{\text{upper}} \approx 9.5 \text{MPa}$, thanks to inertia effects. Inertia controls the void expansion process at a large extent and the $P - a/a_0$ curves for the purely elastic material and the rate-dependent material become similar.

Next, we analyze the role played by the initial void radius $(a_0)$ in the cavity expansion process. While the quasi-static case is not sensitive to the initial dimensions of the cavity [10], inertial resistance to motion increases with the void size [46]. The (full) viscoelastic constitutive model is used in the analysis. Fig. 5 shows the applied pressure $P$ versus $a/a_0$ for three different initial void radii: $a_0 = 0.01 \text{mm}$ (green), $a_0 = 0.1 \text{mm}$ (blue) and $a_0 = 1 \text{mm}$ (black). Results are shown for two different applied pressure rates: $P = 10^5 \text{MPa/s}$ (solid line) and $P = 5 \times 10^5 \text{MPa/s}$ (dotted line). These pressure rates (the highest investigated in Fig. 4) favour that inertia effects play a role in the cavity expansion process. The $P - a/a_0$ curves are always above the upper bound solution $P_{\text{upper}}$. Increasing $a_0$ has the same effect on the $P - a/a_0$ curve that increasing $P$. As the initial cavity radius increases, the
pressure at the cavity wall and the (normalized) size of the void at the time of failure also do.

5.2. Loading condition 2: constant pressure

The pressure rises from zero to $P_0$ at a given rate $P_\dot{a}$ and then remains constant during the rest of the loading process. Fig. 6 shows the growth rate of the void $\dot{a}$ versus the normalized void radius $a/a_0$ for different values of $P_0$, $P_\dot{a}$ and $a_0$. The symbol * indicates the first occurrence of failure at the cavity wall. The (full) viscoelastic constitutive model is used in the calculations. Recall that, due to viscosity, the cavity cannot find an equilibrium configuration for applied pressures larger than $P_{\text{lower}}$. 

- **Fig. 6 (a)** considers $P_0 = 10$ MPa, $a_0 = 0.1$ mm and four different values of $P_\dot{a}$: $10^4$, $10^5$, $10^6$ and $10^7$ MPa/s. The case of $P_\dot{a} \to \infty$ is also taken into account. 
- **Fig. 6 (b)** considers $P_\dot{a} = 10^7$ MPa/s, $a_0 = 0.1$ mm and four different values of $P_0$: 4, 6, 8 and 10 MPa. 
- **Fig. 6 (c)** considers $P_0 = 10$ MPa, $P_\dot{a} = 10^7$ MPa/s and three different values of $a_0$: 0.01, 0.1 and 1 mm.

**Fig. 6.** Growth rate of the void $\dot{a}$ versus the normalized void radius $a/a_0$. (a) We consider $P_0 = 10$ MPa, $a_0 = 0.1$ mm and four different values of $P_\dot{a}$: $10^4$, $10^5$, $10^6$ and $10^7$ MPa/s. The case of $P_\dot{a} \to \infty$ is also taken into account. (b) We consider $P_\dot{a} = 10^7$ MPa/s, $a_0 = 0.1$ mm and four different values of $P_0$: 4, 6, 8 and 10 MPa. (c) We consider $P_0 = 10$ MPa, $P_\dot{a} = 10^7$ MPa/s and three different values of $a_0$: 0.01, 0.1 and 1 mm.

Note that the value of the steady-state expansion velocity increases with

- **Fig. 6 (a)** considers $P_0 = 10$ MPa, $a_0 = 0.1$ mm and four different values of $P_\dot{a}$: $10^4$, $10^5$, $10^6$ and $10^7$ MPa/s. The case of $P_\dot{a} \to \infty$ is also investigated. The value of $P_0$ is larger than the critical equilibrium pressure $P_{\text{lower}}$. For $P_0 = 10^7$ MPa/s and $P_\dot{a} = 10^7$ MPa/s, the expansion velocity is an increasing function of $a/a_0$. As $P_\dot{a}$ increases the growth rate of the cavity $\dot{a}$ also does. The failure of the material occurs before the constant pressure $P_0$ is reached. For these two cases, the loading condition is essentially identical to the one considered in Section 5.1. For $P_\dot{a} = 10^7$ MPa/s, $P = 10^7$ MPa/s and $P_\dot{a} \to \infty$, the $\dot{a} - a/a_0$ curve increases rapidly, reaches a maximum and then decreases slowly. In absence of failure, the expansion velocity of the cavity would reach a constant value (horizontal asymptote in the graph) which identifies the steady-state cavitation regime [37]. The value of the steady-state expansion velocity is determined by $P_0$ and does not depend on $P_\dot{a}$.

- **Fig. 6 (b)** considers $P_\dot{a} = 10^7$ MPa/s, $a_0 = 0.1$ mm and four different values of $P_0$: 4, 6, 8 and 10 MPa. Recall that all the values of $P_0$ are larger than the critical equilibrium pressure $P_{\text{lower}}$. For $P_0 = 4$ MPa, $P_0 = 6$ MPa and $P_0 \to \infty$, the $\dot{a} - a/a_0$ curves show an oscillatory response during the first stages of loading. The value of $\dot{a}$ turns from positive to negative (and vice versa) several times. The amplitude and velocity of the oscillations is gradually reduced until the steady-state cavitation regime ($\dot{a} = \text{constant}$) is reached. For $P_0 = 8$ MPa a single loop is observed in the graph and $\dot{a}$ only takes negative values within a small range of the ratio $a/a_0$. The size of the cavity increases, almost, during the whole loading process. For $P_0 = 10$ MPa the response of the elastic ball is not oscillatory. No exceptions, the cavity size increases monotonically during loading.

Note that the value of the steady-state expansion velocity increases with
The rise of the growth rate of the cavity with the applied pressure boosts the contribution of inertia effects to the expansion process.

Fig. 6(c) considers $P_0 = 10$ MPa, $\dot{P} = 10^7$ MPa/s and three different values of $a_0$: 0.01, 0.1 and 1 mm. The growth rate of the void first increases, reaches and maximum and then gradually decreases until $\dot{a}$ becomes constant. Note that drop between the maximum and the steady value of $\dot{a}$ increases with the void size. Moreover, the growth rate of the void in the steady-state regime significantly increases with the void size. The increase of the cavitation velocity with the void size leads to larger contribution of inertia effects to the expansion process.

Fig. 7 shows the failure time $t_f$ versus the applied pressure rate $\dot{P}$ for (a) purely elastic and (b) viscoelastic materials. Four different values of $P_0$ are considered: 4, 6, 8 and 10 MPa. Three different values of $a_0$ are considered: 0.01, 0.1 and 1 mm.

Irrespective of $P_0$, the $t_f - \dot{P}$ curves are rather similar. For the lowest values of $P$ considered, the failure occurs before the constant value of $P_0$ is reached and $t_f$ decreases linearly with $P$. For the greatest values of $P$ explored the failure occurs after the constant value of $P_0$ is reached and $t_f$ becomes largely independent of $P$. As previously shown in Fig. 6(a), the pressure rate used to reach $P_0$ barely affects the cavity expansion process, including the failure time, in the regime of constant pressure. Moreover, irrespective of the value of $P_0$ considered, the size of the cavity delays failure. The delay is mild when the failure occurs before the cavity pressure becomes constant, but it is very significant when failure occurs during the regime of constant cavity pressure. It is apparent that, for the loading condition investigated in this section of the paper, the influence of inertia effects in the expansion process becomes especially relevant when the cavity pressure is constant.

Fig. 7(b) shows results for the (full) viscoelastic constitutive model. For all the cases analyzed, the failure time is greater than the one corresponding to the purely elastic counterpart. The difference is especially significant in the loading cases in which inertia effects are less important, i.e. low values of $a_0$ and $P_0$. In other words, the
stabilizing effect of viscosity is exposed as long as inertia effects do not control the loading process. The results for $a_0 = 0.01$ mm and $a_n = 0.1$ mm are practically identical for all the values of $P_\infty$ and $P$ explored. Furthermore, the results for $P_\infty = 4$ MPa are virtually independent of $a_0$ and $P$. In this case, unlike what happened for the purely elastic material, the failure occurs after the pressure has reached the constant cavity pressure.

6. Conclusions

In this paper we have conducted a comprehensive finite element analysis to identify the roles of viscosity and inertia in the dynamic expansion of a spherical void embedded into a deformable ball and subjected to internal pressure. The ball is modelled with a nonlinear viscoelastic constitutive theory which incorporates material failure. Numerical simulations, in which the viscosity of the constitutive model has been alternatively switched on and off, have been performed for different loading rates, applied pressures and void sizes. If the pressure at the cavity wall is greater than the critical equilibrium pressure, has been alternatively switched on and off, and have been performed for different loading rates, applied pressures and void sizes. If the pressure at the cavity wall is greater than the critical equilibrium pressure, stabilizes the material behaviour and delay failure. In a general manner, viscous effects are important as long as the cavity pressure does not exceed the upper bound of the rate-dependent material response, i.e. inertia effects become meaningful after the cavity pressure exceeds the upper bound of the rate-dependent material response. Nevertheless, the specific contribution of inertia and viscous effects to the cavity expansion process is highly dependent on the void size. Inertia effects are significantly more important as the cavity size increases.

All in all, this research has shown the need of including viscous and inertia effects in the analysis of elastomers subjected to dynamic loading conditions. This is a key outcome since elastomers are currently widely used in tires, isolation bearings, shock absorbers and many other applications in which they are frequently subjected to shocks, blasts and impacts. In this regard, the prospective work is to extend the application of the viscoelastic constitutive model used in this paper to the aforementioned engineering problems and identify/quantify the actual contribution of viscous and inertia effects to the performance of elastomeric structures subjected to various kinds of dynamic loads.

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